C.d.L. in Ingegneria Elettronica e delle Telecomunicazioni
Corso di Metodi Matematici e Probabilistici

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PROVA SCRITTA

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ESERCIZIO 1 (**punti 5**):

Risolvere la seguente equazione differenziale:

$$x^2y' + y^2 + xy + x^2 = 0 \; .$$

SOLUZIONE:

$$y = \frac{x(1 - \log|Cx|)}{\log|Cx|} \; .$$

ESERCIZIO 2 (**punti 5**):

Risolvere il seguente PVI:

$$\frac{d}{dx}\mathbf{y} = \mathbb{A}\mathbf{y} \, , \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

dove

$$\mathbb{A} = \begin{pmatrix} 0 & 2 \\ \\ -2 & 0 \end{pmatrix}$$

e la funzione incognita $\mathbf{y} : \mathbb{R} \longrightarrow \mathbb{R}^2$.

SOLUZIONE:

$$\mathbf{y} = \begin{pmatrix} \cos 2x + 2\sin 2x \\ -\sin 2x + 2\cos 2x \end{pmatrix} .$$

ESERCIZIO 3 (**punti 5**):

Si consideri la funzione

$$f(x) = \begin{cases} x+1\,, & x \in [-1,0] \\ \\ -2x+1\,, & x \in (0,+1] \end{cases}$$

Se ne tracci il grafico e se ne calcoli lo sviluppo in serie di Fourier come funzione di periodo 2, i.e.

$$f(x+2) = f(x)\,, \quad \forall x \in \mathbb{R}\,.$$

SOLUZIONE:

$$f(x) \sim \frac{1}{4} + \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\pi(2n-1)x}{(2n-1)^2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\pi n x}{n}\,.$$

ESERCIZIO 4 (**punti 5**):

Sia X una v.a. normale, tale che

$$\mathbb{P}(X \le 3) = \mathbb{P}(X \ge -1) = 0.84134\,.$$

Calcolare il valore atteso $\mathbb{E}[X] = \mu$ e la varianza $\mathrm{Var}[X] = \sigma^2$.

SOLUZIONE:

Passiamo dalla v.a. X alla v.a. standardizzata Z. Le ipotesi assegnate possono essere calcolate in termini di Z:

$$0.84134 = \mathbb{P}(X \le 3) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{3-\mu}{\sigma}\right) = \mathbb{P}\left(Z \le \frac{3-\mu}{\sigma}\right)\,,$$

quindi

$$\frac{3-\mu}{\sigma} = z_{0.84134} = 1 \text{ (dalle tavole della normale)}\,.$$

Analogomente per l'altra ipotesi

$$0.84134 = \mathbb{P}(X \ge -1) = \mathbb{P}\left(\tfrac{X-\mu}{\sigma} \ge \tfrac{-1-\mu}{\sigma}\right) = \mathbb{P}\left(Z \ge -\tfrac{1+\mu}{\sigma}\right)$$

$$= 1 - \Phi\left(-\tfrac{1+\mu}{\sigma}\right) = \Phi\left(\tfrac{1+\mu}{\sigma}\right)\,,$$

da cui

$$\frac{1+\mu}{\sigma} = z_{0.84134} = 1\,.$$

Abbiamo quindi

$$\frac{3-\mu}{\sigma} = \frac{1+\mu}{\sigma} = 1$$

e

$$\mathbb{E}[X] = 1 \quad \text{e} \quad \mathrm{Var}[X] = 2\,.$$

ESERCIZIO 5 (**punti 5**):

Consideriamo due v.a. binomiali

$$X_1 \sim \mathcal{B}(10, p) \quad \text{e} \quad X_2 \sim \mathcal{B}(12, p)\,.$$

Si chiede di calcolare (in funzione di p):

a) $\mathbb{P}(X_1 = 10)$,

b) $\mathbb{P}(X_2 = 11)$ ed in questo caso calcolare il valore di p che rende massima la probabilità .

SOLUZIONE:

a) $\mathbb{P}(X_1 = 10) = \binom{10}{10} p^{10}(1-p)^0 = p^{10}$;

b) $\mathbb{P}(X_2 = 11) = \binom{12}{11} p^{11}(1-p) = 12(p^{11} - p^{12})$. Per il calcolo del massimo basta derivare rispetto a p la precedente espressione per ottenere $p = \frac{11}{12}$.

ESERCIZIO 6 (**punti 5**):

Sia X una v.a. di densità

$$f_X(x) = \begin{cases} 1 - |x|\,, & x \in [-1, 1]\,, \\ 0 & \text{altrove}\,. \end{cases}$$

Si chiede:

a) verificare che la $f_X(x)$ sia definita correttamente;

b) calcolare $\mathbb{P}\left(2X^2 \le X\right)$;

c) determinare la funzione di ripartizione $F_X(x)$.

SOLUZIONE:

a) per la verifica, calcoliamo $\int_{-1}^{1}(1-|x|)\,dx = 2\int_0^1 (1-x)\,dx = 2\left(x - \frac{x^2}{2}\right|_0^1 = 1$;

b) $\mathbb{P}\left(2X^2 \le X\right) = \mathbb{P}\left(X(2X-1) \le 0\right)$. Ma la disequazione $X(2X-1) \le 0$ in $[-1, 1]$ ha come unica soluzione $X \in \left[0 \le X \le \frac{1}{2}\right]$ e quindi

$$\mathbb{P}\left(2X^2 \le X\right) = \int_0^{\frac{1}{2}} (1-x)\,dx = \frac{3}{8}\,;$$

c) La funzione di ripartizione $F_X(x)$ vale ovviamente $0\ \forall x \leq -1$ e $1\ \forall x \geq 1$. Inoltre $\forall x \in [-1,0]$ avremo

$$F_X(x) = \int_{-1}^{x} (1+s)ds = \frac{(1+x)^2}{2}$$

e $\forall x \in [0,1]$ avremo

$$F_X(x) = \int_{-1}^{x} (1+|s|)ds = \int_{-1}^{0} (1+s)ds + \int_{0}^{x} (1-s)ds = 1 - \frac{(1-x)^2}{2}\,.$$

In conclusione

$$F_X(x) = \begin{cases} 0\,, \quad x \leq -1 \\ \frac{(1+x)^2}{2}\,,\ x \in [-1,0] \\ 1 - \frac{(1-x)^2}{2}\,,\ x \in [0,1] \\ 1\,, \quad x \geq 1 \end{cases}$$

Tavole della funzione di ripartizione della variabile Normale Standardizzata:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

z	Seconda cifra decimale di z									
	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

z	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.891	4.417
$\Phi(z)$	0.90	0.95	0.975	0.99	0.995	0.999	0.9995	0.99995	0.999995
$2[1-\Phi(z)]$	0.20	0.10	0.05	0.02	0.01	0.002	0.001	0.0001	0.00001