Gradient, Divergence, and Curl Operators

Rectangular coordinates:

$$\nabla\Phi = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Cylindrical coordinates:

$$\nabla\Phi = \left[\hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}\right]$$

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right]$$

Spherical coordinates:

$$\nabla\Phi = \left[\hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right]$$

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$

$$+ \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

VECTOR IDENTITIES

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla\cdot(\nabla\times\mathbf{A})=0$$

$$\nabla\times(\nabla f)=0$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$$
$$+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+(\mathbf{A}\cdot\nabla)f$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})$$
$$+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla\times(f\mathbf{A})=\nabla f\times\mathbf{A}+f\nabla\times\mathbf{A}$$

$$(\nabla\times\mathbf{A})\times\mathbf{A}=(\mathbf{A}\cdot\nabla)\mathbf{A}-\tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A})$$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

INTEGRAL THEOREMS

Line Integral of a Gradient

$$\int_a^b \nabla f\cdot\mathbf{dl}=f(b)-f(a)$$

Divergence Theorem:

$$\int_V \nabla\cdot\mathbf{A}\,dV=\oint_S \mathbf{A}\cdot\mathbf{dS}$$

Corollaries

$$\int_V \nabla f\,dV=\oint_S f\,\mathbf{dS}$$

$$\int_V \nabla\times\mathbf{A}\,dV=-\oint_S \mathbf{A}\times\mathbf{dS}$$

Stokes' Theorem:

$$\oint_L \mathbf{A}\cdot\mathbf{dl}=\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS}$$

Corollary

$$\oint_L f\mathbf{dl}=-\int_S \nabla f\times\mathbf{dS}$$

B59,80

MAXWELL'S EQUATIONS

	Integral	*Differential*	*Boundary Conditions*
Faraday's Law	$\oint_L \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\mathbf{n} \times (\mathbf{E}_2' - \mathbf{E}_1') = 0.$
Ampere's Law with Maxwell's Displacement Current Correction	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$
Gauss's Law	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f dV$	$\nabla \cdot \mathbf{D} = \rho_f$	$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f$
	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$
Conservation of Charge	$\oint_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho_f dV = 0$	$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0$	$\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0$

Usual Linear Constitutive Laws

$\mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \mathbf{E}'$ [Ohm's law for moving media with velocity **v**]

PHYSICAL CONSTANTS

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979 \times 10^8 \approx 3 \times 10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m^2/(kg)^2
Acceleration of gravity	g	9.807	m/(sec)^2
Permittivity of free space	ε_0	$8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$	$376.73 \approx 120\pi$	ohms
Avogadro's number	N	6.023×10^{23}	atoms/mole

B. CURL, DIVERGENCE, AND GRADIENT IN CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

In Secs. 3.1, 3.4, and 9.1 we introduced the curl, divergence, and gradient, respectively, and derived the expressions for them in the Cartesian coordinate system. In this appendix we shall derive the corresponding expressions in the cylindrical and spherical coordinate systems. Considering first the cylindrical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at point $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at point $Q(r + dr, \phi + d\phi, z + dz)$ is as shown in Fig. B.1.

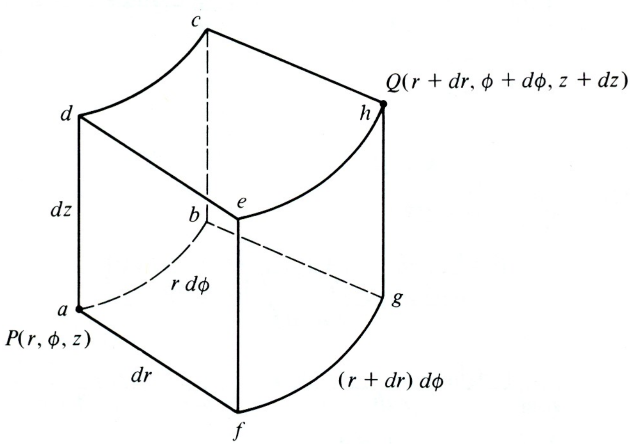

Figure B.1. Infinitesimal box formed by incrementing the coordinates in the cylindrical coordinate system.

From the basic definition of the curl of a vector introduced in Sec. 3.3 and given by

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{i}_n \tag{B.1}$$

we find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Fig. B.1:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\oint_{abcda} \mathbf{A} \cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\phi \to 0 \\ dz \to 0}} \frac{\begin{Bmatrix} [A_\phi]_{(r,z)} r\,d\phi + [A_z]_{(r,\phi+d\phi)}\,dz \\ -[A_\phi]_{(r,z+dz)} r\,d\phi - [A_z]_{(r,\phi)}\,dz \end{Bmatrix}}{r\,d\phi\,dz} \\
&= \lim_{d\phi \to 0} \frac{[A_z]_{(r,\phi+d\phi)} - [A_z]_{(r,\phi)}}{r\,d\phi} + \lim_{dz \to 0} \frac{[A_\phi]_{(r,z)} - [A_\phi]_{(r,z+dz)}}{dz} \\
&= \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}
\end{aligned} \tag{B.2a}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_\phi &= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef} \\
&= \lim_{\substack{dz \to 0 \\ dr \to 0}} \frac{\begin{Bmatrix} [A_z]_{(r,\phi)}\,dz + [A_r]_{(\phi,z+dz)}\,dr \\ -[A_z]_{(r+dr,\phi)}\,dz - [A_r]_{(\phi,z)}\,dr \end{Bmatrix}}{dr\,dz} \\
&= \lim_{dz \to 0} \frac{[A_r]_{(\phi,z+dz)} - [A_r]_{(\phi,z)}}{dz} + \lim_{dr \to 0} \frac{[A_z]_{(r,\phi)} - [A_z]_{(r+dr,\phi)}}{dr} \\
&= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}
\end{aligned} \tag{B.2b}$$

$$\begin{aligned}
(\nabla \times \mathbf{A})_z &= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb} \\
&= \lim_{\substack{dr \to 0 \\ d\phi \to 0}} \frac{\begin{Bmatrix} [A_r]_{(\phi,z)}\,dr + [A_\phi]_{(r+dr,z)}(r+dr)\,d\phi \\ -[A_r]_{(\phi+d\phi,z)}\,dr - [A_\phi]_{(r,z)} r\,d\phi \end{Bmatrix}}{r\,dr\,d\phi} \\
&= \lim_{dr \to 0} \frac{[rA_\phi]_{(r+dr,z)} - [rA_\phi]_{(r,z)}}{r\,dr} + \lim_{d\phi \to 0} \frac{[A_r]_{(\phi,z)} - [A_r]_{(\phi+d\phi,z)}}{r\,d\phi} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}
\end{aligned} \tag{B.2c}$$

Combining (B.2a), (B.2b) and (B.2c), we obtain the expression for the curl of a vector in cylindrical coordinates as

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{i}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\mathbf{i}_\phi + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\mathbf{i}_z = \begin{vmatrix} \frac{\mathbf{i}_r}{r} & \mathbf{i}_\phi & \frac{\mathbf{i}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \tag{B.3}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector, introduced in Sec. 3.6 and given by

$$\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{B.4}$$

Evaluating the right side of (B.4) for the box of Fig. B.1, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \lim_{\substack{dr \to 0 \\ d\phi \to 0 \\ dz \to 0}} \frac{\left\{\begin{array}{l}[A_r]_{r+dr}(r+dr)\,d\phi\,dz - [A_r]_r r\,d\phi\,dz + [A_\phi]_{\phi+d\phi}\,dr\,dz \\ - [A_\phi]_\phi\,dr\,dz + [A_z]_{z+dz} r\,dr\,d\phi - [A_z]_z r\,dr\,d\phi\end{array}\right\}}{r\,dr\,d\phi\,dz} \\ &= \lim_{dr \to 0} \frac{[rA_r]_{r+dr} - [rA_r]_r}{r\,dr} + \lim_{d\phi \to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r\,d\phi} \\ &\quad + \lim_{dz \to 0} \frac{[A_z]_{z+dz} - [A_z]_z}{dz} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \end{aligned} \tag{B.5}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in cylindrical coordinates,

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z \tag{B.6}$$

and hence

$$\begin{aligned} d\Phi &= \frac{\partial \Phi}{\partial r}\,dr + \frac{\partial \Phi}{\partial \phi}\,d\phi + \frac{\partial \Phi}{\partial z}\,dz \\ &= \left(\frac{\partial \Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}\mathbf{i}_\phi + \frac{\partial \Phi}{\partial z}\mathbf{i}_z\right) \cdot (dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z) \\ &= \nabla\Phi \cdot d\mathbf{l} \end{aligned} \tag{B.7}$$

Thus

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{i}_z \tag{B.8}$$

Turning now to the spherical coordinate system, we recall from Appendix A that the infinitesimal box defined by the three orthogonal surfaces intersecting at $P(r, \theta, \phi)$ and the three orthogonal surfaces intersecting at $Q(r + dr, \theta + d\theta, \phi + d\phi)$ is as shown in Fig. B.2. From the basic definition of the curl of a vector given by (B.1), we then find the components of $\nabla \times \mathbf{A}$ as follows with the aid of Fig. B.2:

$$\begin{aligned}
(\nabla \times \mathbf{A})_r &= \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\oint_{abcda} \mathbf{A}\cdot d\mathbf{l}}{\text{area } abcd} \\
&= \lim_{\substack{d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{matrix}[A_\theta]_{(r,\phi)} r\,d\theta + [A_\phi]_{(r,\theta+d\theta)} r\sin(\theta+d\theta)\,d\phi \\ -[A_\theta]_{(r,\phi+d\phi)} r\,d\theta - [A_\phi]_{(r,\theta)} r\sin\theta\,d\phi\end{matrix}\right\}}{r^2\sin\theta\,d\theta\,d\phi} \\
&= \lim_{d\theta\to 0} \frac{[A_\phi\sin\theta]_{(r,\theta+d\theta)} - [A_\phi\sin\theta]_{(r,\theta)}}{r\sin\theta\,d\theta} \\
&\quad + \lim_{d\phi\to 0} \frac{[A_\theta]_{(r,\phi)} - [A_\theta]_{(r,\phi+d\phi)}}{r\sin\theta\,d\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}
\end{aligned} \tag{B.9a}$$

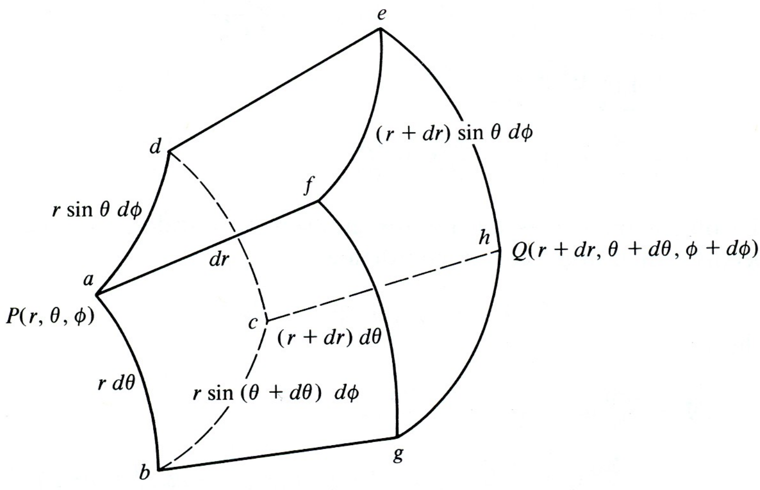

Figure B.2. Infinitesimal box formed by incrementing the coordinates in the spherical coordinate system.

$$(\nabla \times \mathbf{A})_\theta = \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\oint_{adefa} \mathbf{A} \cdot d\mathbf{l}}{\text{area } adef}$$

$$= \lim_{\substack{d\phi \to 0 \\ dr \to 0}} \frac{\left\{ \begin{array}{l} [A_\phi]_{(r,\theta)} r \sin\theta \, d\phi + [A_r]_{(\theta,\phi+d\phi)} \, dr \\ - [A_\phi]_{(r+dr,\theta)} (r + dr) \sin\theta \, d\phi - [A_r]_{(\theta,\phi)} \, dr \end{array} \right\}}{r \sin\theta \, dr \, d\phi}$$

$$= \lim_{d\phi \to 0} \frac{[A_r]_{(\theta,\phi+d\phi)} - [A_r]_{(\theta,\phi)}}{r \sin\theta \, d\phi}$$

$$+ \lim_{dr \to 0} \frac{[rA_\phi]_{(r,\theta)} - [rA_\phi]_{(r+dr,\theta)}}{r \, dr}$$

$$= \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r}(rA_\phi) \tag{B.9b}$$

$$(\nabla \times \mathbf{A})_\phi = \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\oint_{afgba} \mathbf{A} \cdot d\mathbf{l}}{\text{area } afgb}$$

$$= \lim_{\substack{dr \to 0 \\ d\theta \to 0}} \frac{\left\{ \begin{array}{l} [A_r]_{(\theta,\phi)} \, dr + [A_\theta]_{(r+dr,\phi)} (r + dr) \, d\theta \\ - [A_r]_{(\theta+d\theta,\phi)} \, dr - [A_\theta]_{(r,\phi)} r \, d\theta \end{array} \right\}}{r \, dr \, d\theta}$$

$$= \lim_{dr \to 0} \frac{[rA_\theta]_{(r+dr,\phi)} - [rA_\theta]_{(r,\phi)}}{r \, dr}$$

$$+ \lim_{d\theta \to 0} \frac{[A_r]_{(\theta,\phi)} \, dr - [A_r]_{(\theta+d\theta,\phi)} \, dr}{r \, d\theta}$$

$$= \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \tag{B.9c}$$

Combining (B.9a), (B.9b), and (B.9c), we obtain the expression for the curl of a vector in spherical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] \mathbf{i}_r$$

$$+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi) \right] \mathbf{i}_\theta + \frac{1}{r} \left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta} \right] \mathbf{i}_\phi$$

$$= \begin{vmatrix} \dfrac{\mathbf{i}_r}{r^2 \sin\theta} & \dfrac{\mathbf{i}_\theta}{r \sin\theta} & \dfrac{\mathbf{i}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin\theta A_\phi \end{vmatrix} \tag{B.10}$$

To find the expression for the divergence, we make use of the basic definition of the divergence of a vector given by (B.4) and by evaluating its right side for the box of Fig. B.2, we obtain

$$\nabla \cdot \mathbf{A} = \lim_{\substack{dr\to 0\\ d\theta\to 0\\ d\phi\to 0}} \frac{\left\{\begin{matrix}[A_r]_{r+dr}(r+dr)^2 \sin\theta\, d\theta\, d\phi - [A_r]_r r^2 \sin\theta\, d\theta\, d\phi \\ + [A_\theta]_{\theta+d\theta} r \sin(\theta + d\theta)\, dr\, d\phi - [A_\theta]_\theta r \sin\theta\, dr\, d\phi \\ + [A_\phi]_{\phi+d\phi} r\, dr\, d\theta - [A_\phi]_\phi r\, dr\, d\theta\end{matrix}\right\}}{r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

$$= \lim_{dr\to 0} \frac{[r^2 A_r]_{r+dr} - [r^2 A_r]_r}{r^2\, dr} + \lim_{d\theta\to 0} \frac{[A_\theta \sin\theta]_{\theta+d\theta} - [A_\theta \sin\theta]_\theta}{r \sin\theta\, d\theta}$$

$$+ \lim_{d\phi\to 0} \frac{[A_\phi]_{\phi+d\phi} - [A_\phi]_\phi}{r \sin\theta\, d\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{B.11}$$

To obtain the expression for the gradient of a scalar, we recall from Appendix A that in spherical coordinates,

$$d\mathbf{l} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi \tag{B.12}$$

and hence

$$d\Phi = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi$$

$$= \left(\frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi\right)\cdot(dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi)$$

$$= \nabla\Phi \cdot d\mathbf{l} \tag{B.13}$$

Thus

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi \tag{B.14}$$