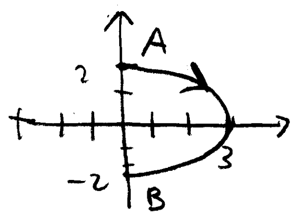
3 punti
3 punti } 6 punti

FILA A (1) Calcolare $\int_\gamma y\,dx$

dove γ è la curva descritta da $\frac{x^2}{9}+\frac{y^2}{4}=1$, $x\geq 0$, percorsa dal punto $(0,2)$ verso il punto $(0,-2)$,
1) Mediante la definizione
2) usando il teorema di Gauss-Green.

1)



$-\gamma: \quad \underline{r}(t)=(3\cos t,\ 2\sin t)$
$t\in[-\pi/2,\ \pi/2]$
$\underline{r}'(t)=(-3\sin t,\ 2\cos t)$

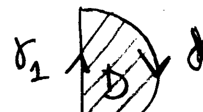
$$\Rightarrow \int_\gamma y\,dx = -\int_{-\pi/2}^{\pi/2} 2\sin t\,(-3\sin t)\,dt$$

$$= 6\int_{-\pi/2}^{\pi/2} \sin^2 t\,dt$$. Essendo $\sin^2 t = \frac{1-\cos(2t)}{2}$ si ha:

$$= 3\int_{-\pi/2}^{\pi/2}[1-\cos(2t)]\,dt = 3\left(t-\frac{\sin(2t)}{2}\right)\Big|_{-\pi/2}^{\pi/2} =$$

$$= 3\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 3\pi$$

2) Per poter utilizzare Gauss-Green devo "chiudere" la curva. Considero γ_1: $\underline{r}_1(t)=(0,t)$, $-2\leq t\leq 2$.
$\Rightarrow \underline{r}_1'(t)=(0,1)$



$\Gamma=\gamma_1\cup(-\gamma)$ curva chiusa semplice, percorsa in senso negativo. Allora:

$$-\int_\Gamma y\,dx = \iint_D (-1)\,dx\,dy$$

Infatti $\underline{F}=(y,0) \Rightarrow \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} = -1$

Risulta:

$$\iint_D (-1)\,dx\,dy = -\text{Area}(D) = -\frac{\pi \cdot 3 \cdot 2}{2} = -3\pi$$

(area ellisse $= \pi a b$ dove a, b lunghezza semiassi)

$$-\int_\Gamma y\,dx = -\int_{\gamma_1} y\,dx - \int_\gamma y\,dx = -3\pi$$

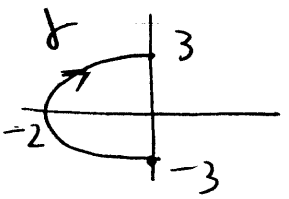
$$\Rightarrow \int_\gamma y\,dx = -\int_{\gamma_1} y\,dx + 3\pi$$

Poiché:

$$\int_{\gamma_1} y\,dx = \int_{-2}^{2} t \cdot 0\,dt = 0$$

risulta $\int_\gamma y\,dx = 3\pi$.

FILA B



$\gamma: \frac{x^2}{4} + \frac{y^2}{9} = 1, \quad x \leq 0$

da $(0,-3)$ a $(0,3)$

$-\gamma: \underline{r}(t) = (2\cos t, 3\sin t), \quad \pi/2 \leq t \leq 3\pi/2$

$\underline{r}'(t) = (-2\sin t, 3\cos t)$

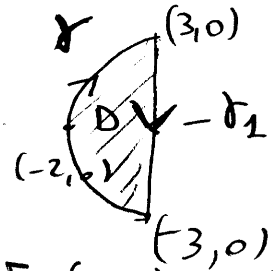
$$\int_\gamma x\,dy = -\int_{\pi/2}^{3\pi/2} 2\cos t \cdot 3\cos t\,dt = -6\int_{\pi/2}^{3\pi/2} \cos^2 t\,dt =$$

$$= -6\int_{\pi/2}^{3\pi/2} \frac{1+\cos 2t}{2}\,dt = -3\left(t + \frac{\sin 2t}{2}\right)_{\pi/2}^{3\pi/2} =$$

$$= -3\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) = \boxed{-3\pi}$$

Con Gauss-Green: $-\gamma_1: \underline{r}_1(t) = (0, t), \quad -3 \leq t \leq 3$

$\underline{r}_1'(t) = (0, 1)$



$\Gamma = \gamma \cup (-\gamma_1)$ percorsa in senso negati

$-\Gamma = -\gamma \cup \gamma_1$ percorsa in senso positivo

Essendo $\underline{F} = (0, x) \Rightarrow \partial F_2/\partial x - \partial F_1/\partial y = 1$

$$\int_{-\Gamma} x\,dy = \iint_D 1\,dx\,dy = \text{Area } D = \frac{6\pi}{2} = 3\pi$$

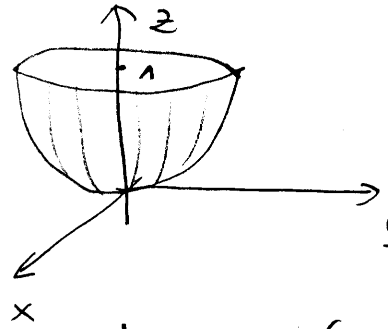
$$\int_{-\Gamma} x\,dy = \int_{-\gamma} x\,dy + \int_{\gamma_1} x\,dy = 3\pi$$

$$\Rightarrow \int_{\gamma} x\,dy = \int_{\gamma_1} x\,dy - 3\pi = -3\pi$$

Infatti $\int_{\gamma_1} x\,dy = \int_{-3}^{3} 0\,dt = 0$

FILA A   5 punti

(2) Sia data la superficie $\Sigma' = \{(x,y,z) : z = x^2+y^2,\ z \le 1\}$
Dopo aver disegnato Σ', calcolare:
$$\iint_{\Sigma'} \frac{|x|}{\sqrt{z}}\, dS.$$



Σ' è la superficie di un paraboloide di rotazione, con asse l'asse z

Parametrizzo Σ' come superficie cartesiana:

$S(x,y) = (x, y, x^2+y^2)$, $(x,y) \in D$
$D = \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2 \le 1\}$
Vettore normale: $\underline{N} = (-2x, -2y, 1)$
$\Rightarrow dS = |\underline{N}|\, dx\, dy = \sqrt{1+4x^2+4y^2}\, dx\, dy$

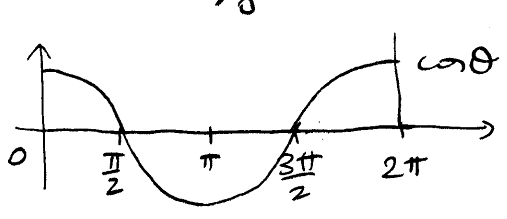
$$\Rightarrow \iint_{\Sigma'} \frac{|x|}{\sqrt{z}}\, dS = \iint_D \frac{|x|}{\sqrt{x^2+y^2}} \sqrt{1+4x^2+4y^2}\, dx\, dy$$

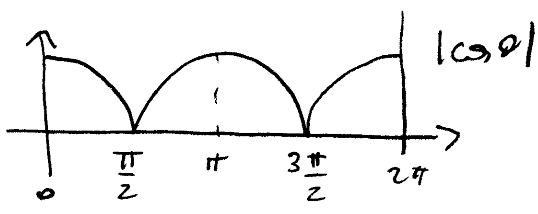
passando a coordinate polari:

$$= \int_0^{2\pi}\int_0^1 \frac{\rho|\cos\theta|}{\rho} \sqrt{1+4\rho^2}\, \rho\, d\rho\, d\theta =$$

$$= \int_0^{2\pi} |\cos\theta|\, d\theta \int_0^1 \rho\sqrt{1+4\rho^2}\, d\rho$$

Poiché: $|\cos\theta|$ è periodico di periodo π





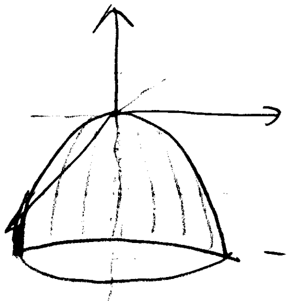
$$= 4\int_0^{\pi/2} \cos\theta\, d\theta \cdot \int_0^1 8\rho\sqrt{1+4\rho^2}\, d\rho \cdot \frac{1}{8} =$$

$$= \frac{4}{8}\left[\sin\theta\right]_0^{\pi/2} \cdot \left[(1+4\rho^2)^{3/2} \cdot \frac{2}{3} \right]_0^1 = \frac{1}{3} \cdot 1 \left(5\sqrt{5}-1\right)$$

$$= \boxed{\frac{5\sqrt{5}-1}{3}}$$

FILA B

$$\Sigma_1 = \{ (x,y,z) : z = -(x^2+y^2),\ z \geq -1 \}$$



$$\iint_{\Sigma_1} \frac{|y|}{\sqrt{-z}}\, dS$$

$$\sigma(x,y) = (x, y, -x^2-y^2), \quad (x,y) \in D$$

$$D = \{ (x,y) : x^2+y^2 \leq 1 \}$$

$$\underline{N} = (2x, 2y, 1)$$

$$\iint_{\Sigma} \frac{|y|}{\sqrt{-z}}\, dS = \iint_D \frac{|y|}{\sqrt{x^2+y^2}} \sqrt{1+4x^2+4y^2}\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \frac{\rho|\sin\theta|}{\rho} \sqrt{1+4\rho^2}\, \rho\, d\rho\, d\theta$$

$$= 4\int_0^{\pi/2} \sin\theta\, d\theta \cdot \int_0^1 \sqrt{1+4\rho^2}\, \rho\, d\rho$$

$$= -4\left[\cos\theta\right]_0^{\pi/2} \left[(1+4\rho^2)^{3/2} \right]_0^1 \cdot \frac{2}{3} \cdot \frac{1}{8}$$

$$= \frac{1}{3}\left(5\sqrt{5}-1\right).$$