MULTIPLE SOLUTIONS FOR ELLIPTIC PROBLEMS

GABRIELE BONANNO
UNIVERSITY OF MESSINA, ITALY

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On the Number of Solutions of Asymptotically Superlinear Two Point Boundary Value Problems

HERBERT AMANN

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Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems

MICHAEL G. CRANDALL & PAUL H. RABINOWITZ

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Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems*

ANTONIO AMBROSETTI

Scuola Normale Superiore, 56100 Pisa, Italy

HAIM BREZIS

Université Paris VI, place Jussieu, 75252 Paris, France; and Rutgers University, New Brunswick, New Jersey 08903

AND

GIOVANNA CERAMI

Università di Palermo, 90100 Palermo, Italy

AMANN

$$(P_\lambda)\begin{cases} -u'' = \lambda f(u) & \text{in} \quad]0,1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is a continuous and nonnegative function such that

$$f(0) > 0; \tag{1}$$

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{2}$$

Then, there is $\bar{\lambda} > 0$ such that the problem (P_λ) has at least two positive solutions for $0 < \lambda < \bar{\lambda}$; at least one for $\lambda = \bar{\lambda}$; none for $\lambda > \bar{\lambda}$.

CRANDALL-RABINOWITZ

$$
(P_\lambda)\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}
$$

$f : \mathbb{R} \to \mathbb{R}$ *is a continuous, nonnegative and sub-critical function such that*

$$f(0) > 0; \qquad \textbf{1}$$

$$m > 2 \quad l > 0 \quad 0 < mF(\xi) \le \xi f(\xi) \quad \textit{for all } \xi \ge l \qquad \textbf{AR}$$

Then, there is $\bar{\lambda} > 0$ *such that the problem* (P_λ) *has at least two positive solutions for* $0 < \lambda < \bar{\lambda}$; *at least one for* $\lambda = \bar{\lambda}$; *none for* $\lambda > \bar{\lambda}$.

The (AR) condition

$$m > 2 \quad l > 0 \qquad 0 < mF(\xi) \le \xi f(\xi) \quad \textit{for all } \xi \ge l$$ **AR**

is a bit more strong than

$$\lim_{t\to+\infty} \frac{f(t)}{t^q} = +\infty \qquad q>1$$ **2′**

that is, f is more than superlinear at infinity.

The condition f(0)>0 implies

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$$ **1′**

that is, f is sublinear at zero.

Condition 1′ *can be true even if* $f(0)=0$, *while if* $f(0)>0$ *then* 0 *is not a solution of the problem.*

So, roughly speaking, Theorem of Amann ensures two positive solutions if

f is more than sublinear at zero

and it is superlinear at infinity,

while Theorem of Crandall-Rabinowitz ensures two positive solutions if

f is more than sublinear at zero

and it is more than superlinear at infinity.

In both cases 0 is not a solution of the problem.

AMBROSETTI-BREZIS-CERAMI

$$(P_\mu)\begin{cases} -\Delta u = \mu u^s + u^q & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where 0<s<1<q, with q subcritical (or critical).

Then, there is $\Lambda > 0$ *such that the problem* (P_μ) *has at least two positive solutions for* $0 < \mu < \Lambda$; *at least one for* $\mu = \Lambda$; *none for* $\mu > \Lambda$.

In this case

$$f(u) = \mu u^s + u^q$$

for which

0 is a solution of the problem

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$$

that is, f is sublinear at zero

$$\lim_{t\to +\infty} \frac{f(t)}{t^q} = +\infty \text{ , } q>1$$

that is, f is more than superlinear at infinity

The aim of this talk is to present an existence result of two positive solutions for the previous problems by requiring, besides the (AR) condition, a condition which is more general than the sublinearity at zero. Precisely, in the ordinary case, we require:

$$\textit{there are two positive constants } \gamma, \delta, \textit{ with } \delta < \gamma \textit{ such that}$$

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{4}\frac{F(\delta)}{\delta^2}. \qquad \mathbf{1''}$$

Function F is the primitive of f. So, in particular,

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty \qquad \textit{implies} \qquad \mathbf{1''}$$

In addition, it may be satisfied also in some case where the functions f are superlinear (or linear) at zero.

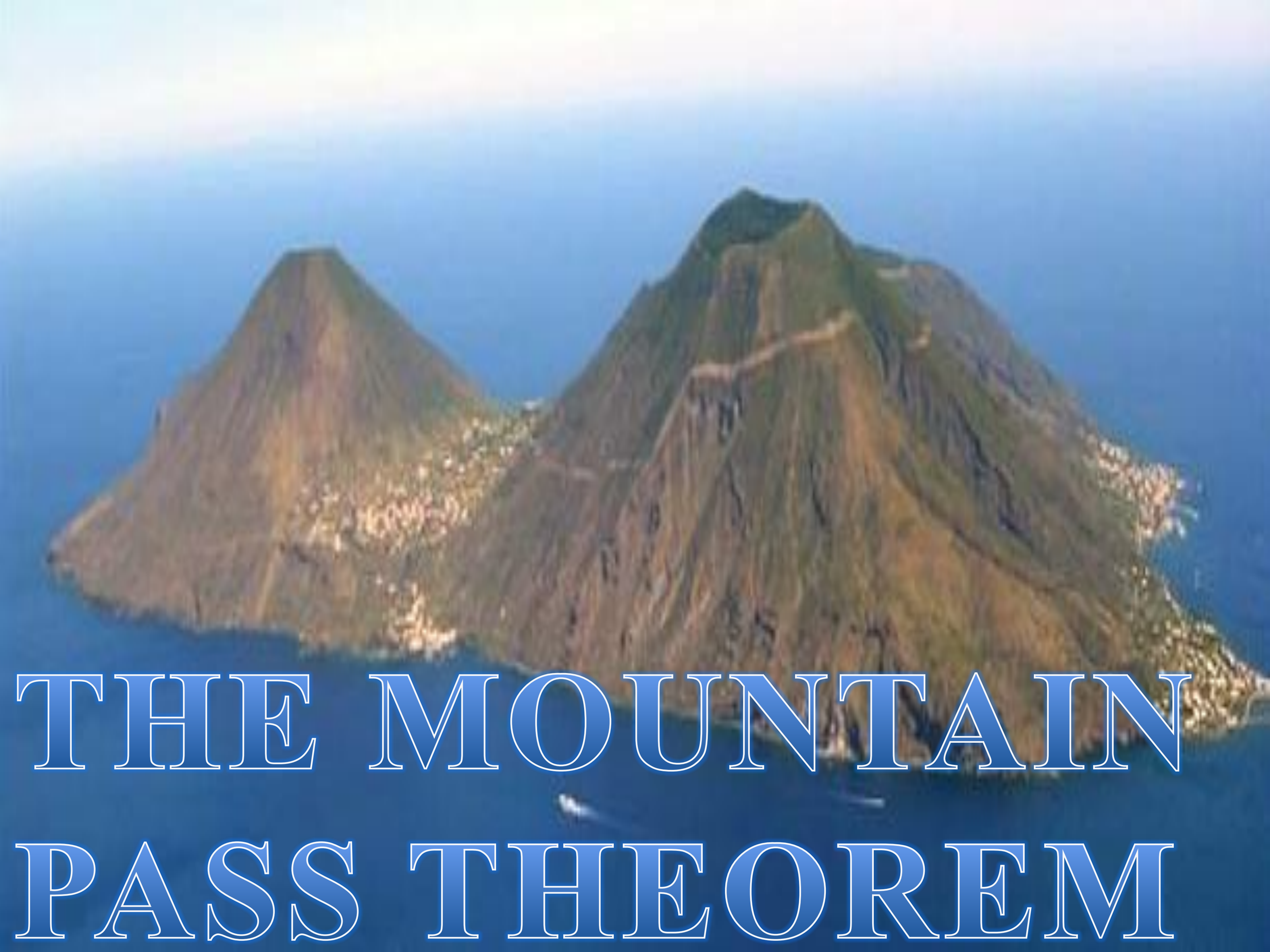
A similar situation one has for elliptic case. In this case such a condition is a bit less simple.

The basic ingredients of such a result are:

a theorem of local minimum

and

the Ambrosetti-Rabinowitz theorem.



THE MOUNTAIN PASS THEOREM

HISTORICAL NOTES

Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).

THE AMBROSETTI-RABINOWITZ THEOREM

1973

Assume that

(G) *there are* $u_0, u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.$$

Then, I *admits a critical value* c *characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.$$

THE PUCCI-SERRIN THEOREM

1985

(G') *there are* $u_0, u_1 \in X$ *and* $r, R \in \mathbb{R}$, *with* $0 < r < R < \|u_1 - u_0\|$, *such that*

$$\inf_{r<\|u-u_0\|<R} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

Corollary. *If I admits two local minima, then I admits a third critical point.*

THE GHOUSSOUB-PREISS THEOREM

1989

(MG) *there are* $u_0, u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

Some remarks on the classical Ambrosetti-Rabinowitz theorem are presented. In particular, it is observed that the geometry of the mountain pass, if the function is bounded from below, is equivalent to the existence of at least two local minima, while, when the function is unbounded from below, it is equivalent to the existence of at least one local minimum.

So, the Ambrosetti-Rabinowitz theorem actually ensures three or two distinct critical points, according to the function is bounded from below or not.

A REMARK ON THE MOUNTAIN PASS GEOMETRY

Theorem. *Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, the following assertions are equivalent:*

(MG) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(L) *I admits at least two distinct local minima.*

A REMARK ON THE MOUNTAIN PASS GEOMETRY

- *So, the Ambrosetti-Rabinowitz theorem, when the function is bounded from below actually ensures three distinct critical points.*

- *In fact, in this case the mountain pass geometry implies the existence of two local minima and the Pucci-Serrin theorem ensures the third critical point.*

A REMARK ON THE MOUNTAIN PASS GEOMETRY

- *In a similar way it is possible to see that, when the function is unbounded from below, the mountain pass geometry is equivalent to the existence of at least one local minimum.*

In this case, the following condition is requested:

The function I is bounded from below on every bounded set of X.

A REMARK ON THE MOUNTAIN PASS GEOMETRY

REMARK *Let X be a real Banach space and $I : X \to \mathbb{R}$ be a functional of class C^1 satisfying the $(PS)-$condition and the mountain pass geometry (MG). Assume that I is bounded from below on every bounded set of X. Then, I admits two or three distinct critical points according to whether it is unbounded from below or not.*

BONANNO G., *A characterization of the mountain pass geometry for functionals bounded from below,* Differential and Integral Equations **25** (2012), 1135-1142.

A LOCAL MINIMUM THEOREM

Our aim is to present a local minimum theorem for functionals of the type:

Φ $-$ Ψ

$$\Phi - \Psi$$

A LOCAL MINIMUM THEOREM

It is an existence theorem of a critical point for continuously Gâteaux differentiable functions, possibly unbounded from below.

The approach is based on Ekeland's Variational Principle applied to a non-smooth variational framework by using also a novel type of Palais-Smale condition which is more general than the classical one.

BONANNO G., *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992-3007.

BONANNO G., *Relations between the mountain pass theorem and local minima,* Advances in Nonlinear Analysis, **1** (2012), 205-220.

A LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \qquad (1)$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \qquad (2)$$

Moreover, assume that I satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'(u_0) = 0$.

A LOCAL MINIMUM THEOREM

Three consequences

First

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with Φ bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi < +\infty$ and assume that,

for each $\lambda \in \left] 0, \dfrac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *the functional*

$I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(PS)^{[r]}$*-condition.*

Then, for each $\lambda \in \left] 0, \dfrac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *there is*

$u_1 \in \Phi^{-1}(]-\infty, r[)$ *such that* $I_\lambda(u_1) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]-\infty, r[)$ *and* $I'_\lambda(u_1) = 0$.

A LOCAL MINIMUM THEOREM

Three consequences

Second

Let X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions. Fix* $\inf_X \Phi < r < \sup_X \Phi$ *and assume that*

$$\rho(r) > 0,$$

where $\rho(r) = \sup_{v \in \Phi^{-1}(]r, \infty[)} \dfrac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{\Phi(v) - r}$, *and*

for each $\lambda > \dfrac{1}{\rho(r)}$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is bounded from below and satisfies* ${}^{[r]}$*(PS)-condition.*

Then, for each $\lambda > \dfrac{1}{\rho(r)}$ *there is* $u_2 \in \Phi^{-1}(]r, +\infty[)$ *such that* $I_\lambda(u_2) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r, +\infty[)$ *and* $I'_\lambda(u_2) = 0$.

A LOCAL MINIMUM THEOREM

Three consequences

Third

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition.*

Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *there is $u_0 \in \Phi^{-1}(]0, r[)$ (hence, $u_0 \neq 0$) such that $I_\lambda(u_0) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_0) = 0$.*

We have seen that the mountain pass theorem is actually a theorem of multiplicity. In the sense that

1. *If the functional is bounded from below, then we have at least three critical points;*

2. *If the functional is unbounded from below, then we have at least two critical points.*

Indeed, in such a theorem, one of the key assumptions, that is, the mountain pass geometry is equivalent to the existence of local minima.

Thus, by combining the mountain pass theorem with the local minimum theorem, we get multiple solutions.

The Mountain Pass Theorem + **The Local Minimum Theorem** = **MULTIPLE CRITICAL POINTS**

To be precise, from the mountain pass theorem we obtain the following multiple critical points results:

A Three Critical Points Theorem

by using the first and the second consequence of the local minimum theorem;

A Two Critical Points Theorem

by using the first consequence of the local minimum theorem;

A Two Nonzero Critical Points Theorem

by using the third consequence of the local minimum theorem.

Clearly, since such results are obtained from the mountain pass theorem can happen that in the applications we get results already well-known or results that can be directly obtained by the mountain pass theorem and classical techniques. This can often happen, for instance, in the case of two critical points theorems. While, the situation is different in the cases of three critical points theorem and two nonzero critical points theorem.

Now, we give the statements of such moltiple critical points.

A THREE CRITICAL POINTS THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals with Φ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}. \qquad \text{(G1)}$$

Further assume that, for each

$$\lambda \in \Lambda := \left] \frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.

Then, for each $\lambda \in \Lambda$ the functional I_λ admits at least three critical points.

A THREE CRITICAL POINTS THEOREM

AN EASY EXAMPLE

Consider the following two point boundary value problem

$$(P_\lambda) \quad \begin{cases} -u'' = \lambda f(u) \ \text{ in } \ ]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and λ is a positive real parameter. Put

$$F(\xi) = \int_0^\xi f(t)dt$$

for all $\xi \in \mathbf{R}$ and assume, for clarity, that f is nonnegative.

A THREE CRITICAL POINTS THEOREM
AN EASY EXAMPLE

Theorem. *Assume that*
there are two positive constants c and d, with $c < d$, such that

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2} \qquad (1)$$

and there are two positive constants a and s, with $s < 2$, such that

$$F(\xi) \le a(1 + |\xi|^s) \qquad \forall \xi \in \mathbb{R}. \qquad (2)$$

Then, for each $\lambda \in \left]8\frac{d^2}{F(d)}, 2\frac{c^2}{F(c)}\right[$, problem (P_λ) admits at least three (nonnegative) classical solutions.

A THREE CRITICAL POINTS THEOREM

- *Two-point boundary value problems*
- *Neumann boundary value problems*
- *Mixed boundary value problems*
- *Sturm-Liouville boundary value problems*
- *Hamiltonian Systems*
- *Fourth-order elastic beam equations*
- *Boundary value problems on the half-line*
- *Nonlinear diffence problems*
- *Impulsive equations*
- *Fractional equations*
- *Impulsive fractional equations*

A THREE CRITICAL POINTS THEOREM

- *Elliptic Dirichlet problems involving the p-lapacian with p>n*
- *Elliptic Neumann problems involving the p-laplacian with p >n*
- *Mixed elliptic problems involving the p-laplacian with p >n*
- *Elliptic Systems*
- *Elliptic Dirichlet problems*
- *Elliptic Neumann problems*
- ….
- ….
- ….

A BRIEF NOD TO HISTORY OF THREE CRITICAL POINTS THEOREMS

Papers based on the PUCCI-SERRIN THEOREM

- **RICCERI B.**, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), 220–226.
- **BONANNO G.**, *Some remarks on a three critical points theorem*, Nonlinear Analysis **54** (2003), 651-665.
- **AVERNA D., BONANNO G.,** *A three critical points theorem and its applications to ordinary Dirichlet problem*, Topological Methods in Nonlinear Analysis **22** (2003), 93-103.
- **ARCOYA D., CARMONA J.,** *A nondifferentiable extension of a theoem of Pucci-Serrin and applications*, Journal of Differential Equations, **235** (2007), 683-700.
- **BONANNO G., MARANO S.A.**, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Applicable Analysis, **89** (2010), 1-10.
- **BONANNO G.**, *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992-3007.

Some versions in the non-smooth case

- **BONANNO G., MOTREANU D., WINKERT P.**, *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, Journal of Mathematical Analysis and Application, **381** (2011), 627-637.
- **BONANNO G., WINKERT P.**, *Multiplicity results to a class of variational-hemivariational inequalities*, Topological Methods in Nonlinear Analysis, **43** n.2 (2014), 493–516.
- **BONANNO G., MOTREANU D., WINKERT P.**, *Boundary value problems with nonsmooth potential, constraints and parameters*, Dynamic Systems and Applications **22** (2013), 385-396.

AN EXAMPLE IN NON-SMOOTH CASES

$$\begin{cases} -(p(x)u')' + q(x)u \in \lambda F(u) & \text{in } (a,b) \\ u(a) = u'(b) = 0. \end{cases} \tag{1.1}$$

Theorem 1.1. *Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be u.s.c. with compact convex values, and $\alpha > 0$, $s \in (1,2)$, $0 < c < d$ s.t.*

(*i*) $0 \leqslant \min F(t) \leqslant \alpha(1 + |t|^{s-1})$ *for all* $t \in \mathbb{R}$;

(*ii*) $\displaystyle \frac{1}{c^2}\int_0^c \min F(t)\,dt < \frac{K}{d^2}\int_0^d \min F(t)\,dt$, *with* $K = 3p_0\big(12\|p\|_\infty + 4(b-a)^2\|q\|_\infty\big)^{-1}$.

Moreover, set

$$\Lambda = \left(\frac{p_0 d^2}{2K(b-a)^2}\Big(\int_0^d \min F(t)\,dt\Big)^{-1},\ \frac{p_0 c^2}{2(b-a)^2}\Big(\int_0^c \min F(t)\,dt\Big)^{-1} \right).$$

Then, for all $\lambda \in \Lambda$ problem (1.1) has at least three solutions.

BONANNO G., IANNIZZOTTO A., MARRAS M., *On ordinary differential inclusions with mixed boundary conditions,* Differential and Integral Equations, to appear.

SOME RECENT PAPERS

BONANNO G., LIVREA R., MAWHIN J., *Existence results for parametric boundary value problems involving the mean curvature operator*, Nonlinear Differential Equations and Applications NoDEA, **22** (2015), 411-426.

BARLETTA G., BONANNO G., O'REGAN D., *A variational approach to multiplicity results for boundary value problems on the real line*, Proceedings of the Royal Society of Edinburgh, Section A, **140** (2015), 13-29.

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BONANNO G., D'AGUI' G., PAPAGEORGIOU N.S., *Infinitely many solutions for mixed elliptic problems involving the p-Laplacian*, Advanced in Nonlinear Studies **15** (2015), 939-950**.**

BONANNO G., O'REGAN D., VETRO F., *Sequences of distinct solutions for boundary value problems on the real line*, Journal of Nonlinear and Convex Analysis, **17** n.2 (2016), 365-375.

BONANNO G., CHINNI' A., TERSIAN S., *Existence results for a two point boundary value problem involving a fourth-order equation,* Electronic Journal of Qualitative Theory of Differential Equations, **2015**, n.33 (2015), 1-9.

BONANNO G., DI BELLA B., HENDERSON J., *Infinitely many solutions for a boundary value problem with impulsive effects,* Boundary Value Problems n.278 **2013** (2013), 1-14.

BONANNO G., TORNATORE E., *Existence and multiplicity of solutions for nonlinear elliptic Dirichlet systems,* Electronic Journal of Differential Equations, **2012** n.183 (2012), 1-11.

A TWO CRITICAL POINTS THEOREM

Theorem: *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) < +\infty$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

A TWO CRITICAL POINTS THEOREM

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative, sub-critical continuous function.*

Assume that

$$0 < \mu F(t) \leq t f(t)$$

AR

for all $t \geq r$, for some $r > 0$ and for some $\mu > 2$.

Then, there exists $\lambda^* > 0$ such that for each $\lambda \in]0, \lambda^*[$, the problem

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

admits at least two weak solutions.

A TWO CRITICAL POINTS THEOREM

If in addition we assume that

$$\lim_{t\to 0^+}\frac{f(t)}{t}=0$$

(hence f(0)=0, for which the problem admits the zero solution. Moreover, in this case, one has $\lambda^* = +\infty$ *)*

we obtain the same result of Ambrosetti and Rabinowitz.

If, on the contrary, we assume

$$f(0)>0$$

(that is, $f(0) \neq 0$, for which the problem does not admit the zero solution.)

we obtain a result of type Crandall and Rabinowitz.

A TWO NONZERO CRITICAL POINTS THEOREM

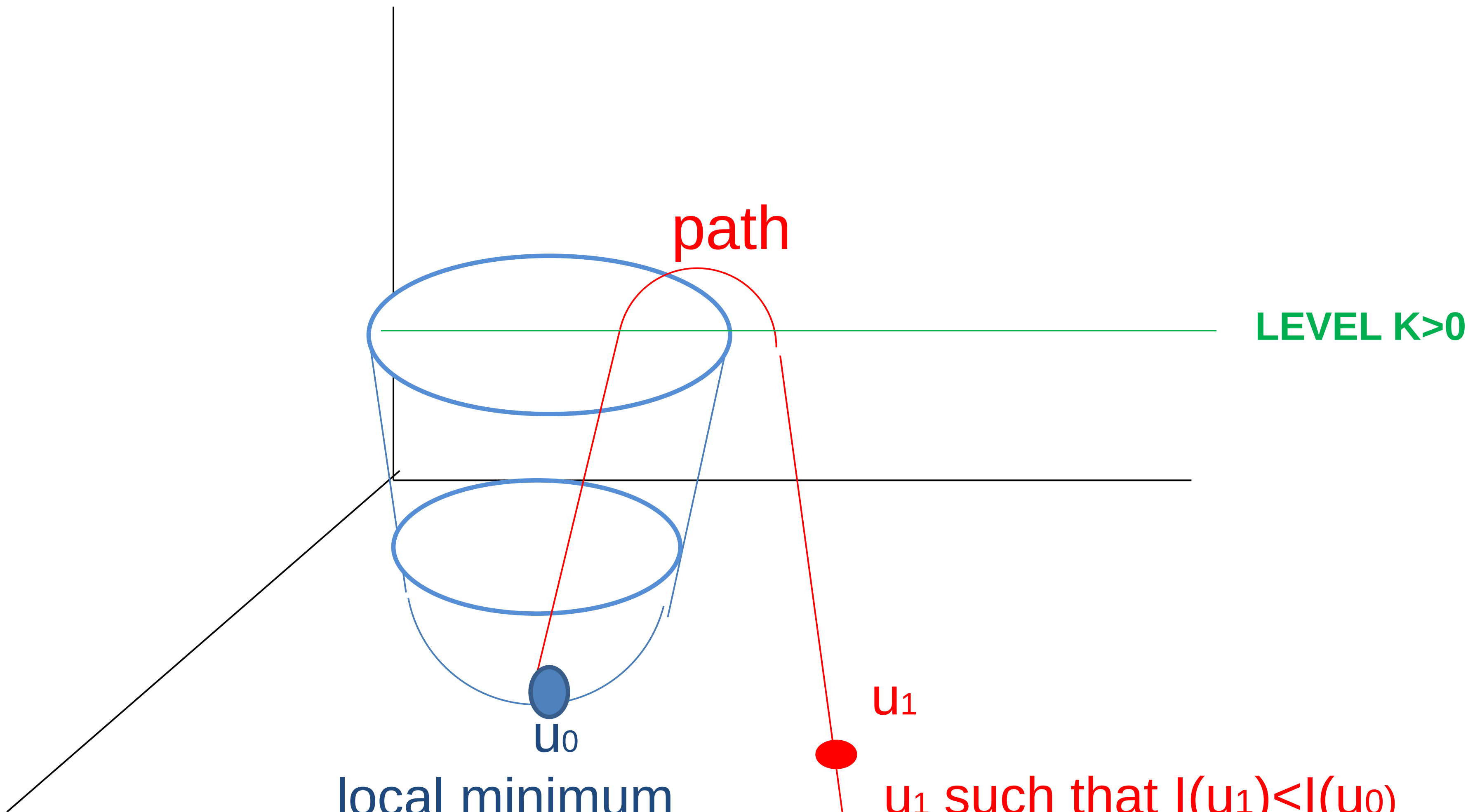
Theorem *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \qquad \text{(G2)}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies (PS)-condition and it is unbounded from below.
Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

A TWO NONZERO CRITICAL POINTS THEOREM

BONANNO G., D'AGUI' G., *Two non-zero solutions for elliptic Dirichlet problems,* Zeitschrift für Analysis und ihre Anwendungen , **35** n.4 (2016), 449-464.



A TWO NONZERO CRITICAL POINTS THEOREM

Remark 2.1. In Theorem 2.1, we can assume the Cerami condition or (C)−condition as introduced by Cerami instead of (PS)−condition, provided that the coercivity of Φ is assumed. The (C)−condition is slightly weaker than (PS)−condition

Consider the problem

$$(P_\lambda)\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function which is nonnegative and continuous in $[0, +\infty[$.

Assume that

(h) *there exist* $s \in [1, 2[$, $q \in]2, 2N/(N-2)[$ *and two positive constants* a_s, a_q *such that*

$$f(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1}$$

for all $t \geq 0$.

A TWO NONZERO CRITICAL POINTS THEOREM

Moreover, put $R(x) = \sup\{\delta : B(x,\delta) \subseteq \Omega\}$ for all $x \in \Omega$, and $R = \sup_{x\in\Omega} R(x)$, for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. Finally, put

$$K = \frac{R^2}{2(2^N - 1)} \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}$$

and

$$\Xi_\delta = \frac{1}{K} \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \frac{\delta^2}{F(\delta)}, \qquad \Lambda_\gamma = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \frac{1}{\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}}$$

where γ, δ are positive constants.

$$T = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma(1+N/2)} \right)^{1/N}$$

$$\|u\|_{L^{2^*}(\Omega)} \leq T\|u\| \quad \forall\, u \in H_0^1(\Omega)$$

A TWO NONZERO CRITICAL POINTS THEOREM

Theorem 3.1. *Assume that* (h) *holds. Moreover, assume that there are two positive constants* γ *and* δ, *with* $\delta < \gamma$, *such that*

$$\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2} < K\frac{F(\delta)}{\delta^2} \tag{3.1}$$

and there are two constants $m > 2$ *and* $l > 0$ *such that, for all* $\xi \geq l$, *one has*

$$0 < mF(\xi) \leq \xi f(\xi). \tag{AR}$$

Then, for each $\lambda \in]\Xi_\delta, \Lambda_\gamma[$, *problem* (P_λ) *admits at least two positive weak solutions.*

$$\lambda^* = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}} \left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}} \left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}} \frac{q-2}{q-s}$$

Corollary 3.1. *Assume* (h),

$$\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty \tag{3.1'}$$

and (AR).

Then, for each $\lambda \in]0, \lambda^*[$, *problem* (P_λ) *admits at least two positive weak solutions.*

A TWO NONZERO CRITICAL POINTS THEOREM

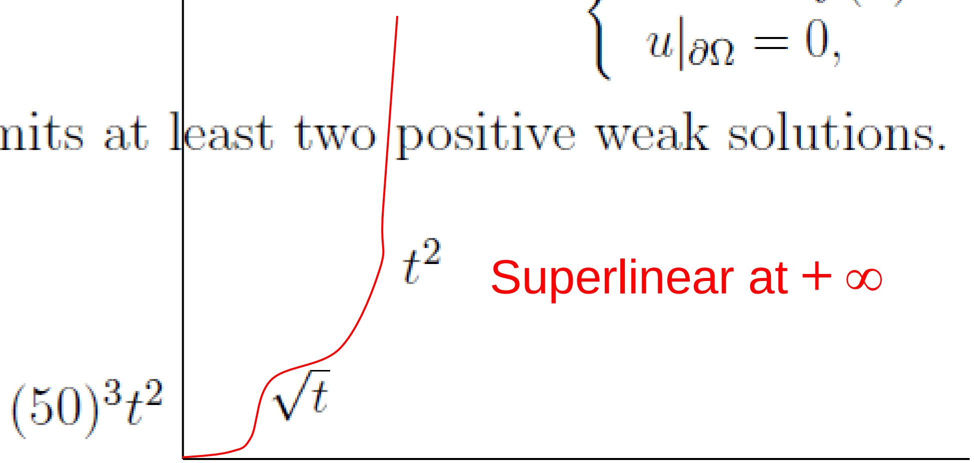
Example 3.1. Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) = \begin{cases} (50)^3 t^2 & \text{if} \quad t \le \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if} \quad \left(\frac{1}{50}\right)^2 < t \le 1, \\ t^2 & \text{if} \quad t \ge 1. \end{cases}$$

Owing to previous theorem, the problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions.



Superlinear at 0

A TWO NONZERO CRITICAL POINTS THEOREM

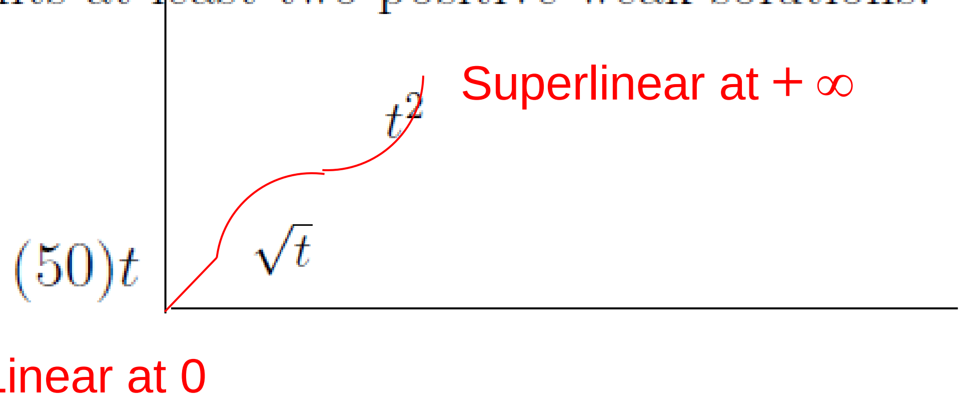
Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) = \begin{cases} (50)t & \text{if} \quad t \leq \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if} \quad \left(\frac{1}{50}\right)^2 < t \leq 1, \\ t^2 & \text{if} \quad t \geq 1. \end{cases}$$

Owing to previous theorem, the problem

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions.



A TWO NONZERO CRITICAL POINTS THEOREM

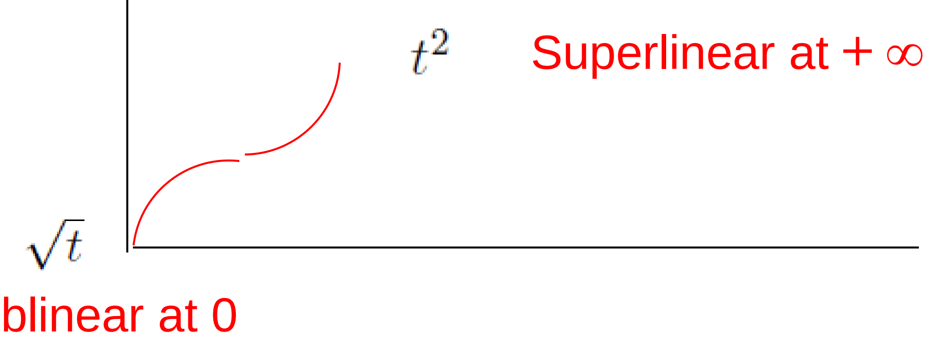
Example 3.2. Owing to Corollary 3.1, for each $\lambda \in \left]0, \frac{\sqrt{3}}{4T^2|\Omega|^{\frac{2}{N}}}\right[$ the problem

$$\begin{cases} -\Delta u = \lambda \max\{\sqrt{u}, u^2\} \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions. Moreover, in particular, if $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$, the problem

$$\begin{cases} -\Delta u = \frac{1}{2} \max\{\sqrt{u}, u^2\} \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions, since $\frac{1}{2} < \lambda^* = \frac{9}{16} \sqrt[6]{3} \left(\frac{\pi}{2}\right)^{\frac{2}{3}}$.



We recall that in the Crandall-Rabinowitz theorem, besides (AR) condition, the key assumption is

$$f(0)>0$$

Hence, the Crandall-Rabinowitz theorem cannot applied to none of previous examples since, there, one has $f(0)=0$.

Moreover, we also observe that

$$f(0) > 0 \Longrightarrow f \text{ sublinear at } 0^+$$

Now, put

$$\mu^* = \left(\frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\right)^{\frac{q-s}{q-2}} s(q-2)q^{\frac{2-s}{q-2}} \left(\frac{(2-s)^{(2-s)}}{(q-s)^{(q-s)}}\right)^{\frac{1}{q-2}}$$

Corollary 3.2. *Fix* $1 \leq s < 2 < q < 2^*$. *Then, for each* $\mu \in]0, \mu^*[$ *problem*

$$(D_\mu) \quad \begin{cases} -\Delta u = \mu u^{s-1} + u^{q-1} & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

admits at least two positive weak solutions.

Indeed, it is enough to observe that

$$\lambda^* = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \left(\frac{s}{\mu}\right)^{\frac{q-2}{q-s}} (q)^{\frac{2-s}{q-s}} \left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}} \frac{q-2}{q-s} > 1.$$

So, from our result applied to $-\Delta u = \lambda \left(\mu u^{s-1} + u^{q-1}\right)$ *we obtain the conclusion.*

Remark 3.3. Corollary 3.2 is a particular case of the very nice result established in the fundamental and seminal work [3] by a clever combination of topological and variational methods. Precisely, in [3] the existence of a first positive solution, by using the method of sub- and super-solutions, is established and then, through a deep reasoning, by proving that this first solution is the minimum of a suitable functional associated to a modified problem, the mountain pass theorem is applied in order to obtain a second positive solution. However, in this type of proof, no numerical estimate of the superior, called Λ, of parameters μ for which the problem (D_μ) admits such solutions is provided. We observe that our proof of Corollary 3.2 is totally variational. Indeed, the first positive solution is directly obtained as a local minimum and the second one is obtained by applying the mountain pass theorem but without modifying the functional in order to establish the positivity of the second solution. In addition, we observe that the same proof of Corollary 3.2 gives precise numerical values μ for which (D_μ) is solved .

A TWO NONZERO CRITICAL POINTS THEOREM

We observe that our results and the Ambosetti-Brezis-Cerami as well as Crandall-Rabinowitz are mutually independent. Indeed, on the hand, we can apply our results to problems where ABC and CR cannot be applied, as seen in the previous examples. On the other hand, when we can apply both ABC and our results, the value Λ obtained in ABC, even if given in a theorical form, is the best. So, in this latest case we can use our results as a complement to ABC in order to give a numerical lower bound of Λ, that is,

$$\mu^* \leq \Lambda.$$

The same remark also for CR holds.

A TWO NONZERO CRITICAL POINTS THEOREM

We observe that, as in recent results, we can use a condition which is a bit more general than the (AR)-condition, by using the (C)-condition instead of (PS)-condition. To be precise, we can assume in our theorem, the following conditions

(b_1) $\displaystyle\lim_{t\to+\infty}\frac{F(t)}{t^2}=+\infty;$

(b_2) *there exist* $\tau\in\left]\min\left\{\frac{(q-2)N}{2},s\right\},\frac{2N}{N-2}\right[$ *and* $\gamma_0>0$ *such that*

$$\liminf_{t\to+\infty}\frac{tf(t)-2F(t)}{t^\tau}\geq\gamma_0;$$

instead of the (AR)-condition. So, our theorem can be stisfied by functions which are only superlinear at + ∞, as for instance, the following function:

A TWO NONZERO CRITICAL POINTS THEOREM

$$f(t) = \begin{cases} (50)t & \text{if} \quad t \leq \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if} \quad \left(\frac{1}{50}\right)^2 < t \leq 1, \\ \dfrac{t}{\log 2}\log(1+t) & \text{if} \quad t \geq 1 \end{cases}$$

Such a function is linear at 0 and only superlinear at $+\infty$.

In any case, the condition (b) does not exhaust the entire class of functions which are superlinear at infinity. For example, the following function

$$f(t) = t^2(\sin t + 2)$$

does not satisfy neither the (b_2)-condition nor the (AR)-condition.

ORDINARY CASE

Previous results also for the ordinary case holds true. However, in this case we can obtain the following more precise result.

Theorem 3.2. *Let $f : [0,+\infty[\to [0,+\infty[$ be a continuous function and assume that* (AR) *holds. Moreover, assume that there are two positive constants γ, δ, with $\delta < \gamma$ such that*

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{4}\frac{F(\delta)}{\delta^2}. \tag{3.1''}$$

Then, for each $\lambda \in \left]\frac{8\delta^2}{F(\delta)}, \frac{2\gamma^2}{F(\gamma)}\right[$, the problem

$$\begin{cases} -u'' = \lambda f(u) \quad \text{in} \quad]0,1[, \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ^f}$$

admits at least two positive classical solutions.

A TWO NONZERO CRITICAL POINTS THEOREM

$$\text{If } \limsup_{\delta\to 0^+} \frac{F(\delta)}{\delta^2} = +\infty \text{ then } (3.1'') \text{ holds true}$$

and in this case the interval becames $\left]0, \bar{\lambda}\right[$, where

$$\bar{\lambda} = \sup_{\gamma>0} \frac{2\gamma^2}{F(\gamma)}.$$

So, in particular, our result holds by assuming, besides the *(AR) – condition*, the following condition

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty.$$

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$f(t) = \begin{cases} t^2 & \text{if} \quad t < 1, \\ \sqrt{t} & \text{if} \quad 1 \leq t < 10^2, \\ \frac{1}{10^3} t^2 & \text{if} \quad t \geq 10^2. \end{cases}$$

Owing to Theorem 3.2, the problem

$$\begin{cases} -u'' = 5^2 f(u) & \text{in} \quad]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions. It is enough to to verify $\frac{1}{2}\frac{F(3)}{3^2} < \frac{1}{5^2} < \frac{1}{8}\frac{F(1)}{1^2}$. We observe that in this case, the nonlinearity f is not sublinear at zero.

Example 3.4. For each $\lambda \in]0,3[$ the problem

$$\begin{cases} -u'' = \lambda \max\{\sqrt{u}, u^2\} & \text{in} \quad]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions.

We recall that in the Amann theorem, besides the condition $\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$ *,*

the key assumption is ***f(0)>0***

Hence, the Amann theorem cannot applied to none of previous examples since, there, one has f(0)=0.
Moreover, we also observe that

$$f(0) > 0 \Longrightarrow f \text{ sublinear at } 0^+$$

A TWO NONZERO CRITICAL POINTS THEOREM

We observe that our results and the Amann theorem are mutually independent. Indeed, on the hand, we can apply our results to problems where the result of Amann cannot be applied, as seen in the previous examples. On the other hand, Amann requires only the superlinearity at infinity and, in addition, when we can apply both the results, the value Λ obtained in Amann, even if given in a theorical form, is the best. So, in this latest case we can use our results as a complement to the result of Amann in order to give a numerical lower bound of Λ, that is,

$$\bar{\lambda} \leq \Lambda.$$

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A TWO NONZERO CRITICAL POINTS THEOREM

BONANNO G., JEBELEAN P., SERBAN C., *Superlinear discrete problems,* Applied Mathematics Letters **52** (2016), 162-168.

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A FINAL REMARK

Applying the Mountain Pass Theorem with the help of the local minimum theorem we obtain, roughly speaking, these two types of results:

f superlinear at zero and f sublinear at infinity →two positive solutions

(by applying the three critical points theorem)

f sublinear at zero and f superlinear at infinity →two positive solutions

(by applying the two non-zero critical points theorem)

A FINAL REMARK

Indeed, we have actually required

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2} \qquad\qquad for \quad some \quad c < d$$

which is more general than the superlinearity at zero, and

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2} \qquad\qquad for \quad some \quad d < c$$

which is more general than the sublinearity at zero.

So in both cases we have no condition at zero, while at $+\infty$, we require either the sublinearity or the superlinearity. It depends, in both cases, by the Palais-Smale condition.

A FINAL REMARK

Is it possible to establish a result of multiple solutions without any conditions at infinity?

To this aim we explicit the following remark on the mountain pass geometry and on the Palais-Smale condition. At first we recall again the mountain pass theorem.

The Mountain Pass Theorem

1. *Mountain pass geometry*
2. *Palais-Smale condition*

Then, there is a critical point.


PATH


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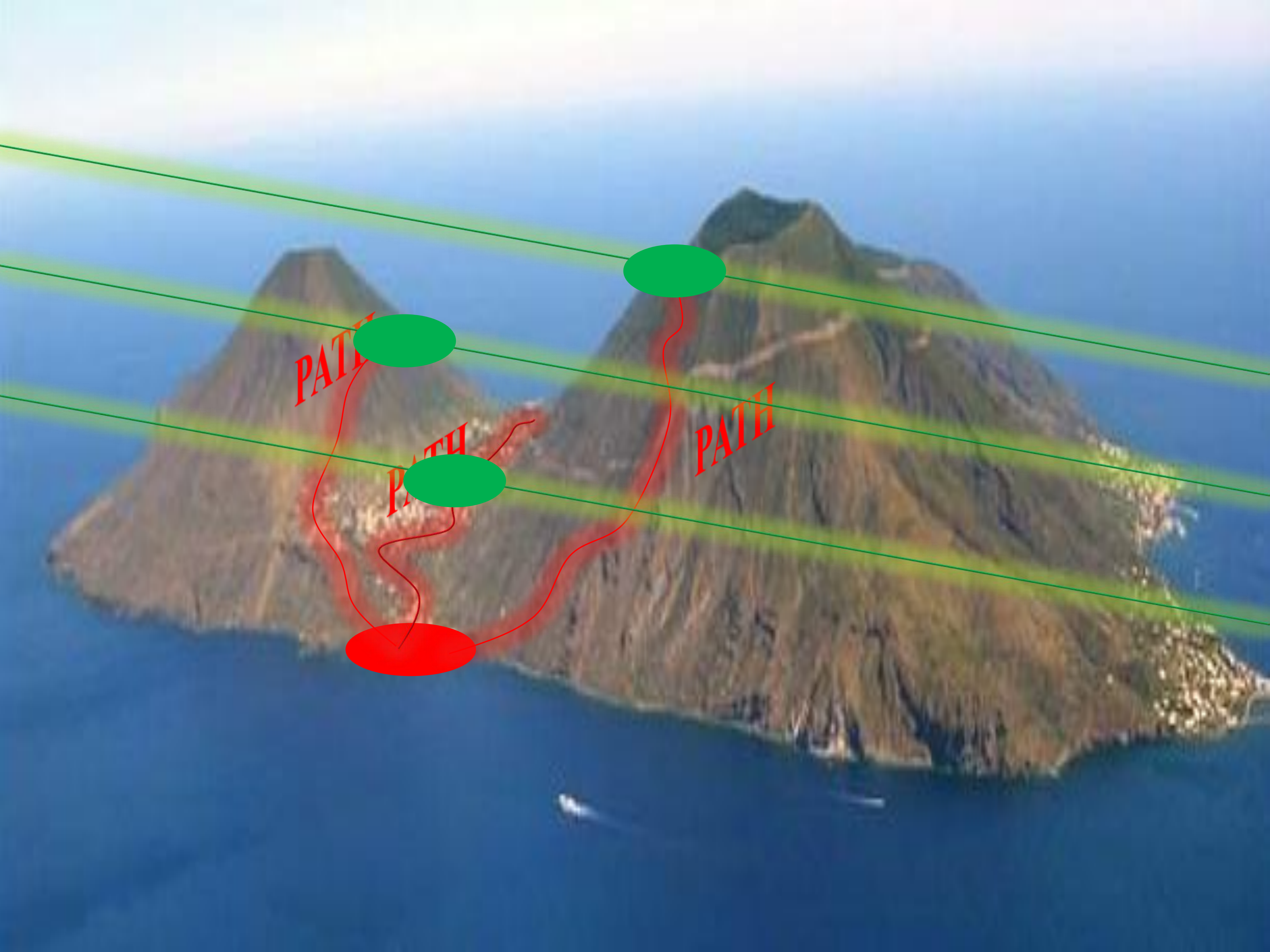
Higher level of the path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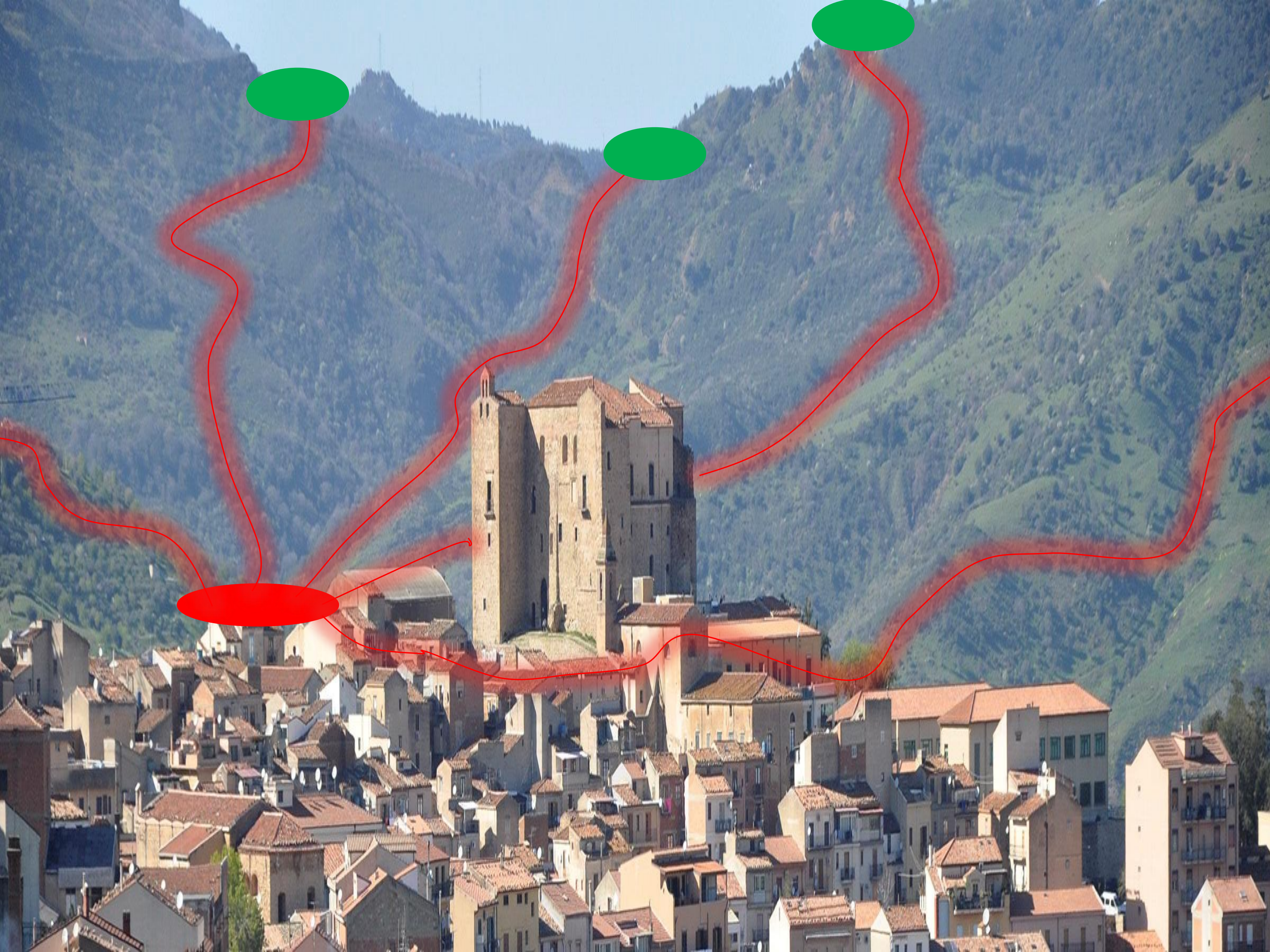
Mountain pass point


Mountain pass point


Mountain pass point



A FINAL REMARK

We have two different structures of the mountains. The structure of Salina which is a general structure and the structure of Castelbuono which is a special structure of the mountains.

Is it possible to give an analytic form to a structure of type Castelbuono?

The answer is positive and the following analytic form expresses a structure of type Castelbuono:

$$\text{there is } r > 0 \text{ such that} \quad \sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) < \frac{r}{2}$$

If we combine this condition with that of the mountain pass geometry, we get a strong mountain pass geometry.

The Mountain Pass Theorem

1. *Mountain pass geometry*
2. *Palais-Smale condition*

Then, there is a critical point.

A special version of the Mountain Pass Theorem

1. *Strong mountain pass geometry*
2. *Weak Palais-Smale condition*

Then, there is a critical point near to the local minimum.

BONANNO G., CANDITO P., *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, Journal of Differential Equations **244** (2008), 3031-3059.

A FINAL REMARK

Corollary 3.1. *Assume that there are two positive constants* ρ_1, ρ_2 *and* $\overline{v} \in X$, *with* $\rho_1 < \Phi(\overline{v}) < \rho_2/2$, *such that*

(a_1'') $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)}\Upsilon(u)}{\rho_1} < \dfrac{1}{2}\dfrac{\Upsilon(\overline{v})}{\Phi(\overline{v})}$;

(a_2') $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)}\Upsilon(u)}{\rho_2} < \dfrac{1}{4}\dfrac{\Upsilon(\overline{v})}{\Phi(\overline{v})}$.

Assume also that for each

$$\lambda \in \Lambda'_{\rho_1,\rho_2,\overline{v}} := \left]\frac{2\Phi(\overline{v})}{\Upsilon(\overline{v})}, \min\left\{\frac{\rho_1}{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)}\Upsilon(u)}; \frac{\rho_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)}\Upsilon(u)}\right\}\right[$$

one has

(b_3) *the functional* $\Phi - \lambda\Upsilon$ *fulfills* $(PS)_c^{\frac{\rho_2}{2\lambda}}$, $c \in \mathbb{R}$.

Then, for each $\lambda \in \Lambda'_{\rho_1,\rho_2,\overline{v}}$ *the functional* I_λ *admits three critical points* u_1, u_2, u_3 *which lie in* $\Phi^{-1}(]-\infty,\rho_2[)$.

BONANNO G., CANDITO P., *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, Journal of Differential Equations **244** (2008), 3031-3059.

A FINAL REMARK

Theorem. *Assume that there are three positive constants c_1, d and c_2, with $c_1 < d < \frac{\sqrt{2}}{2} c_2$, such that*

$$\frac{F(c_1)}{c_1^2} < \frac{1}{6} \frac{F(d)}{d^2}$$

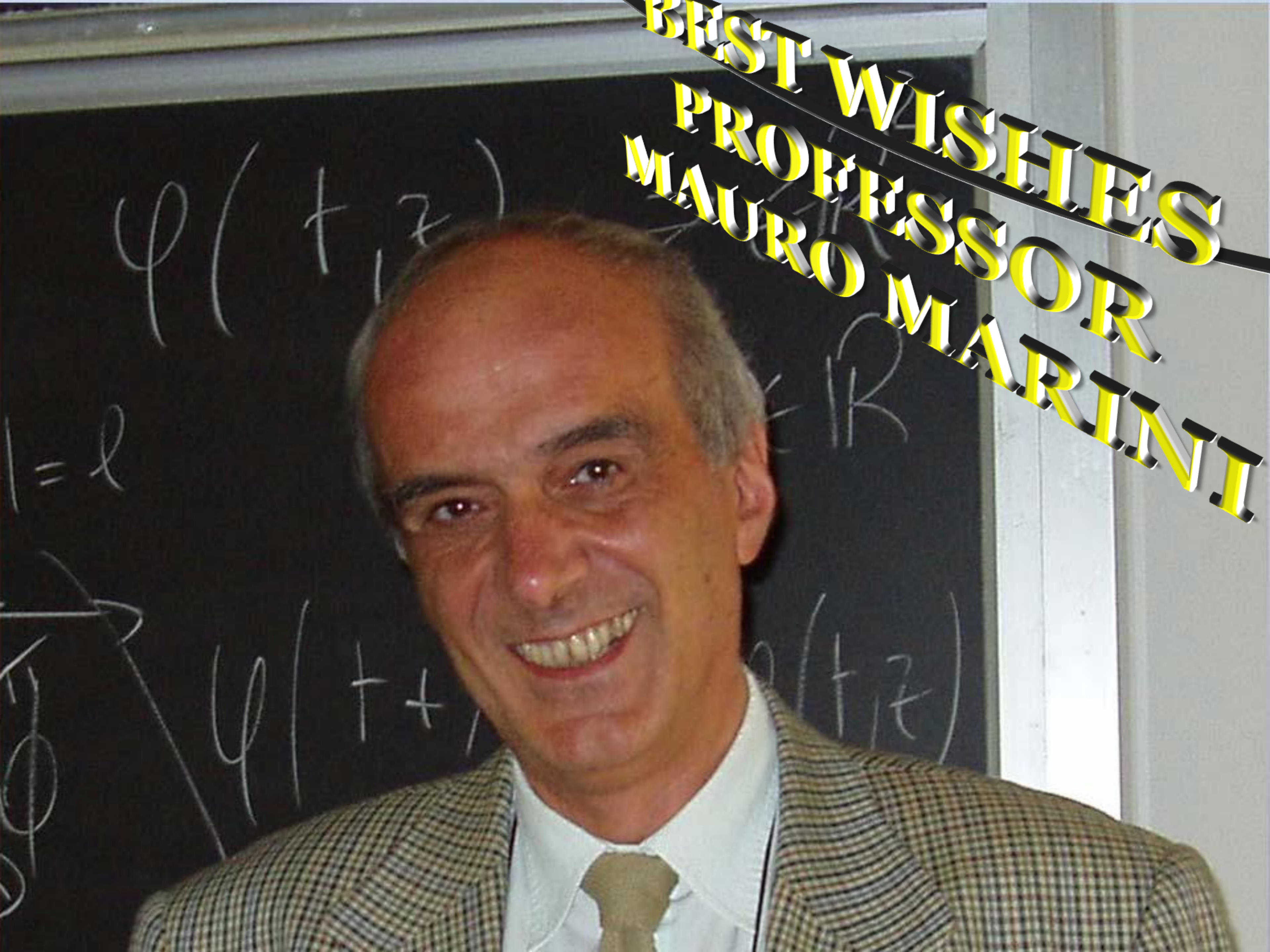
and

$$\frac{F(c_2)}{c_2^2} < \frac{1}{12} \frac{F(d)}{d^2}.$$

Then, for each $\lambda \in \left] 12 \frac{d^2}{F(d)}, \min \left\{ 2 \frac{c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right[$, problem (P_λ) admits at least three (nonnegative) classical solutions u_i, $i = 1, 2, 3$, such that

$$\max_{x \in [0,1]} |u_i(x)| < c_2, \qquad i = 1, 2, 3.$$


Thank you very much for your kind attention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