Asymptotic theory of second order nonlinear differential equations: "quadro completo"

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Joint research with Mauro Marini

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...piccolo bilancio

MathSciNet e Scopus su Mauro Marini:

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La nostra collaborazione comincia nel anni 1992-1993 a Firenze...


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On the Qualitative Behavior of Solutions of Third Order Differential Equations

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1. INTRODUCTION

The purpose of this paper is to examine the oscillatory and nonoscillatory behavior of solutions of the third order linear differential equation

$$\left[\frac{1}{p(t)}\left(\frac{1}{r(t)}x'\right)'\right]' + q(t)x = 0 \qquad \left(' = \frac{d}{dt}\right) \tag{E$_1$}$$

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Main topics of our research:

- **Third order differential equations** (1996–2000) Equivalence theorem on properties A,B.
- **Asymptotic theory for second order nonlinear differential equations** (2000–2016)
- **Higher order differential equations** (2011–2014) Equivalency for disconjugate operators. Oscillation and asymptotics for equations with the middle term.
- **Difference equations** (since 2001) Similarities and discrepancies between continuous and discrete case.
- **Boundary value problems for second order differential equations on the half-line** (since 2011)

... insieme abbiammo publicato 57 lavori.

Motivation: PDE with p-Laplacian

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + B(|x|)F(u) = 0, \quad x \in E,$$

where $E = \{x \in R^n : |x| \geq c > 0\}$. Put $r = |x|$. The function $u = u(x)$ is its radially symmetric solution $\Leftrightarrow$ $y = y(r) = u(|x|)$ is a solution of second order ODE

$$\left(r^{n-1}\Phi(y')\right)' + r^{n-1}B(r)F(y) = 0, \quad (r \geq c). \tag{1}$$

where $\Phi(u) = |u|^{p-2}u$, $p > 1$.

$p = 2$: p-Laplacian $\rightsquigarrow$ Laplacian: $\Delta u = \operatorname{div} \operatorname{grad} u$

$n = 2 : \int_c^\infty \frac{1}{r^{n-1}} dr = \infty \ldots$ Case I
$n = 3 : \int_c^\infty \frac{1}{r^{n-1}} dr < \infty \ldots$ Case II

Let $\alpha = p - 1$. Then (1) can be written as equation

$$\left(a(t)|x'|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0 \tag{E}$$

Assumptions: $a, b \in C[0, \infty)$, $a(t) > 0$ and $b(t) \geq 0$ for $t \geq 0$, $\sup\{b(t) : t \geq T\} > 0$ for any $T \geq 1$, $\alpha > 0$, $\beta > 0$.

Case I: $\displaystyle\int_0^\infty a^{-1/\alpha}(s)ds = \infty, \quad \int_0^\infty b(s)ds < \infty,$

Case II: $\displaystyle\int_0^\infty a^{-1/\alpha}(s)ds < \infty, \quad \int_0^\infty b(s)ds = \infty.$

Two different cases:

- $\alpha = \beta$ (half-linear case)
- $\alpha \neq \beta$ (quasi-linear case)

Historical survey

Emden-Fowler differential equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta \neq 1, \beta > 0. \tag{2}$$

where a, b are positive continuous functions for $t \geq 0$.

$\beta > 1$: super-linear equation, $\beta < 1$: sub-linear equation

- $\hookrightarrow$ Emden (1907): Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und metheorologische Probleme, Leipzig.
- $\hookrightarrow$ Fowler (1930): The solutions of Emden's and similar differential equations, Monthly Notices Roy. Astronom. Soc.
- $\hookrightarrow$ Atkinson (1955), Moore and Nehari (1959) ... $\beta > 1$
- $\hookrightarrow$ Belohorec (1961) ... $\beta < 1$

A solution x is *nonoscillatory* if it has no zero for large t.
In view of the sign of b, all nonoscillatory sol's of (2) satisfy

$$x(t)x'(t) > 0 \quad \text{for large } t.$$

If x is a sol. of (2), then $-x$ is a sol. too. So, we will consider only nonoscillatory solutions which are eventually positive.

Positive solutions can be classified as:

subdominant $\iff$ $x(\infty) = c_x$, $x'(\infty) = 0$,

intermediate $\iff$ $x(\infty) = \infty$, $x'(\infty) = 0$,

dominant $\iff$ $x(\infty) = \infty$, $x'(\infty) = d_x$,

c_x, d_x are positive constants.

If x, y and z are subdominant, intermediate and dominant sols, then

$$0 < x(t) < y(t) < z(t) \quad \text{for large } t.$$

Moore and Nehari (Trans. Amer. Math. J. 1959): $\beta > 1$

- Necessary/sufficient conditions for the existence of dominant solutions of (2)
- Necessary/sufficient conditions for the existence of subdominant solutions of (2)
- The above three types of nonoscillatory sols cannot coexist simultaneously!
- No conditions for the existence of intermediate solutions are given.

Two years later Belohorec proved the same results in the sublinear case, i.e. $\beta < 1$.

Equation with p-Laplacian

Due to the interest for radially symmetric solutions of PDE with p-Laplacian, Kusano and Elbert (1990), Kusano et all (1998), Mirzov (2000)...

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \quad (t \geq 0). \qquad \text{(E)}$$

Assumptions: $\alpha > 0$, $\beta > 0$, $a, b \in C[0,\infty)$, $a(t) > 0$, $b(t) > 0$.

$$I_a = \int_0^{\infty} a^{-1/\alpha}(s)ds = \infty, \quad I_b = \int_0^{\infty} b(s)ds < \infty.$$

For (E) the above classification as subdominant, intermediate and dominant solutions continues to hold by replacing in the Moore and Nehari classification the derivative with the quasiderivative

$$x' \qquad \rightsquigarrow \qquad x^{[1]} = a(t)|x'(t)|^{\alpha}\operatorname{sgn} x'(t).$$

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \qquad \text{(E)}$$

$$J_{\alpha} = \int_0^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_t^{\infty} b(s)\,ds\right)^{1/\alpha} dt,$$

$$K_{\beta} = \int_0^{\infty} b(t) \left(\int_0^{t} \frac{1}{a^{1/\alpha}(s)}\,ds\right)^{\beta} dt.$$

- (E) has subdominant solutions $\iff J_{\alpha} < \infty$.
- (E) has dominant solutions $\iff K_{\beta} < \infty$.

Our target: Quadro completo

- Relations between J_{α} and K_{β}?
- Necessary/sufficient conditions for the existence of intermediate solutions ?

CHANGE OF INTEGRATION FOR J_α, K_β

$\alpha = \beta$: Z.D.- I. Vrkoč (2004),

$\alpha \neq \beta$: M. Cecchi, M. Marini, I. Vrkoč, Z.D., *Integral conditions for nonoscillation of second order nonlinear differential equations*, Nonlinear Anal. 2006.

$\rightsquigarrow$ Compatibility of conditions:

$$J_\alpha = \infty, \quad K_\beta = \infty$$
$$J_\alpha < \infty, \quad K_\beta < \infty$$
$$J_\alpha = \infty, \quad K_\beta < \infty \quad \text{for } \alpha = \beta > 1 \text{ or } \alpha > \beta$$
$$J_\alpha < \infty, \quad K_\beta = \infty \quad \text{for } \alpha = \beta < 1 \text{ or } \alpha < \beta.$$

- If $\alpha \neq \beta$, $J_\alpha = \infty$, $K_\beta = \infty$, then every sol. is oscillatory (Mirzov 1993).

Half-linear equation and "quadro completo"

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\alpha}\operatorname{sgn} x = 0 \qquad \text{(H)}$$

- ↪ M. Cecchi, M. Marini, Z.D.,*Half-linear equations and characteristic properties of the principal solution*, JDE 2005
- ↪ M. Cecchi, M. Marini, Z.D., *On intermediate solutions and the Wronskian for half-linear differential equations*, J. Math. Anal. Appl. 2007

Existence of intermediate solutions:

- Sturm-theory for half-linear equation
- the notion of principal and nonprincipal solutions

Coexistence problem: Is it possible for (H) the coexistence of subdominant, intermediate and dominant solutions?

$$
\begin{aligned}
(C_1):\quad & J_\alpha = \infty,\ K_\alpha = \infty, && \text{if } 0 < \alpha;\\
(C_2):\quad & J_\alpha = \infty,\ K_\alpha < \infty, && \text{if } 1 < \alpha;\\
(C_3):\quad & J_\alpha < \infty,\ K_\alpha = \infty, && \text{if } 0 < \alpha < 1;\\
(C_4):\quad & J_\alpha < \infty,\ K_\alpha < \infty, && \text{if } 0 < \alpha.
\end{aligned}
$$

Theorem 1

If (C_1) *holds and (H) is nonoscillatory, then* $\mathbb{M} = \mathbb{M}_{\infty,0} \neq \emptyset$;

If (C_2) *holds, then* $\mathbb{M}_{\infty,0} \neq \emptyset,\ \mathbb{M}_{\infty,B} \neq \emptyset,\ \mathbb{M}_B = \emptyset$;

If (C_3) *holds, then* $\mathbb{M}_{\infty,0} \neq \emptyset,\ \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty,B} = \emptyset$.

If (C_4) *holds, then* $\mathbb{M}^+_{\infty,\ell} \neq \emptyset,\ \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty 0} = \emptyset$.

Sub-linear equation and "quadro completo"

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0, \quad \alpha > \beta.$$

Existence of intermediate solutions:

$$J_\alpha = \infty,\ K_\alpha < \infty:\ \mathbb{M}_B = \emptyset,\ \mathbb{M}_{\infty B} \neq \emptyset,\ \ \mathbb{M}_{\infty 0} \neq \emptyset$$

Jingfa, Funkc. Equat. (1989) – a priori bounds – fixed point theorem

Coexistence problem: These three types of nonoscillatory sols cannot coexist simultaneously!

$$J_\alpha < \infty,\ K_\beta < \infty:\ \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty B} \neq \emptyset,\ \ \mathbb{M}_{\infty 0} = \emptyset$$

M. Naito J. Math. Anal. Appl. (2011).

Super-linear equation and "quadro completo"

Coexistence problem – partial answer when $0 < \alpha < 1$ (half-linearization method): All three types of nonoscillatory sols cannot coexist simultaneously!

$\hookrightarrow$ M. Cecchi, M. Marini, Z.D., *Intermediate solutions for Emden-Fowler type equations: continuous versus discrete*, Advances Dynam. Systems Appl. (2008).

The difficult problems:

1 Coexistence problem when $\alpha < \beta$, $\alpha > 1$

2 Existence of intermediate solutions when $\alpha < \beta$

Coexistence problem

$$\big(a(t)|x'|^{\alpha}\operatorname{sgn} x'\big)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \quad (t \ge 0) \qquad \text{(E)}$$

where $b(t) \ge 0$ and $\alpha < \beta$.

$\hookrightarrow$ M. Marini, Z.D., *On super-linear Emden-Fowler type differential equations*, J. Math. Anal. Appl.(2014).

Theorem 2

Let $J_\alpha < \infty$ and $K_\beta < \infty$. Then (E) *does not have intermediate solutions.*

Consequently, (E) never has simultaneously subdominant, intermediate and dominant solutions!

This is an extension of Moore-Nehari result for (1).

Idea of the proof.

Step 1. New Holder-type inequality:

Lemma

Let λ, μ be such that $\mu > 1, \lambda\mu > 1$ and let f, g be nonnegative continuous functions for $t \geq T$. Then

$$\left(\int_T^t g(s)\left(\int_s^t f(\tau)d\tau\right)^{\lambda} ds\right)^{\mu}$$

$$\leq K\left(\int_T^t f(\tau)\left(\int_T^{\tau} g(s)ds\right)^{\mu} d\tau\right)\left(\int_T^t f(\tau)d\tau\right)^{\lambda\mu-1}$$

$$K = \lambda^{\mu}\left(\frac{\mu-1}{\lambda\mu-1}\right)^{\mu-1}$$

Step 2. Asymptotic property of intermediate solutions for equation

$$\left(|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0. \tag{E1}$$

Lemma

Let $1 < \alpha < \beta$ *and assume*

$$\int_0^{\infty} s^{\beta} b(s) ds < \infty.$$

Then for any intermediate solution x *of (9) we have*

$$\liminf_{t\to\infty} \frac{tx'(t)}{x(t)} > 0.$$

Step 3. Nonexistence of intermediate solution for (E1). Let $\alpha > 1$, x be an intermediate solution of (E1). Then

$$\lim_{t\to\infty} t^{-1}x(t) = 0 \quad \Longrightarrow \quad x(t) < t \quad \text{large } t.$$

By Step 2,

$$tx'(t) > m\, x(t). \tag{3}$$

Suppose

$$\int_{t_1}^{\infty} s^{\beta} b(s)\, ds < m/2. \tag{4}$$

Integrating (E1) on (T, t)

$$\big(x'(T)\big)^{\alpha} - \big(x'(t)\big)^{\alpha} = \int_T^t b(s)x^{\beta-\alpha}(s)x^{\alpha}(s)ds <$$

$$< \frac{1}{m}\int_T^t b(s)x^{\beta-\alpha}(s)s^{\alpha}\big(x'(s)\big)^{\alpha}ds \le \frac{(x'(T))^{\alpha}}{m}\int_T^t s^{\beta}b(s)ds \le \frac{(x'(T))^{\alpha}}{2}.$$

Thus

$$\frac{(x'(T))^{\alpha}}{2} < \big(x'(t)\big)^{\alpha},$$

which gives a contradiction as $t \to \infty$, since $\lim_{t\to\infty} x'(t) = 0$.

Step 4. Extension to the general weight a:

Set

$$A(t) = \int_0^t a^{-1/\alpha}(\sigma)d\sigma.$$

The change of variable

$$s = A(t), \quad X(s) = x(t), \ t \in [0, \infty), \ s \in [0, \infty)$$

transforms (E), $t \in [0, \infty)$, into

$$\frac{d}{ds}\left(|\dot{X}(s)|^{\alpha} \operatorname{sgn} \dot{X}(s)\right) + c(s)X^{\beta}(s) = 0, \quad s \in [0, \infty),$$

$t(s)$ is the inverse function of $s(t)$, the function c is given by

$$c(s) = a^{1/\alpha}(t(s)))b(t(s)).$$

Existence of intermediate solutions

Consider

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \quad \alpha < \beta. \tag{E}$$

$\hookrightarrow$ M. Marini, Z.D., *Monotonicity conditions in oscillation to super-linear differential equations*, Electron. J. Qual. Theory Differ. Equ.(2015).

$\rightsquigarrow$ Integral conditions in case $\alpha < \beta$

$$
\begin{aligned}
&J_\alpha = \infty, \quad K_\beta = \infty \quad \text{(all sols oscillatory)}\\
&J_\alpha < \infty, \quad K_\beta < \infty : \ \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty B} \neq \emptyset,\ \ \mathbb{M}_{\infty 0} = \emptyset\\
&J_\alpha < \infty, \quad K_\beta = \infty : \ \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty B} = \emptyset,\ \mathbb{M}_{\infty 0} \neq \emptyset??
\end{aligned}
$$

Necessary condition: *If it has intermediate solutions, then*

$$J_\alpha < \infty, \quad K_\beta = \infty. \tag{5}$$

For Emden-Fowler equation

$$x'' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad \beta > 1$$

(5) reads

$$\int_1^\infty t\, b(t)dt < \infty, \quad \int_1^\infty t^\beta\, b(t)dt = \infty.$$

Consider Emden-Fowler equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta > 1 \tag{EF}$$

where $t \geq 1$.

Theorem 3

Let

$$\int_1^{\infty} t\, b(t) dt < \infty, \quad \int_1^{\infty} t^{\beta}\, b(t) dt = \infty$$

and

$$F(t) = t^{(\beta+3)/2} b(t) \text{ be nonincreasing for } t \geq T.$$

Then (EF) has infinitely many intermediate solutions which are positive increasing on $[T, \infty)$.

Idea of the proof.
Define for any solution x of (EF) the energy function

$$E_x(t) = t\left(x'(t)\right)^2 - x(t)x'(t) + \frac{2}{\beta+1} t b(t)|x(t)|^{\beta+1}. \qquad (6)$$

Lemma

If $F(t) = t^{(\beta+3)/2} b(t)$ is nonincreasing for $t \geq T$, then for any solution x

$$\frac{d}{dt} E_x(t) < 0 \quad t \geq T.$$

$$E_x(t) = t\left(x'(t)\right)^2 - x(t)x'(t) + \frac{2}{\beta+1} tb(t)|x(t)|^{\beta+1}.$$

Lemma

If x be a subdominant or oscillatory solution of (2), then

$$\liminf_{t\to\infty} E_x(t) \geq 0.$$

Lemma

Eq. (2) has both solutions x for which

$$E_x(T) < 0$$

at some $T \geq 1$.

Existence of x for which $E_x(T) < 0$: Let x start at

$$x(T) = m > 0, \; x'(T) = \theta.$$

Then

$$E_x(T) = T\left(\theta^2 - \frac{1}{T}m\theta + \frac{2}{\beta+1}b(T)m^{\beta+1}\right).$$

Put

$$f(\theta) = \theta^2 - \frac{1}{T}m\theta + \frac{2}{\beta+1}b(T)m^{\beta+1},$$

m is a positive parameter. Let θ_1, θ_2 be the positive zeros of f, $\theta_1 < \theta < \theta_2$ and

$$m < \left(\frac{1}{T^2}\frac{\beta+1}{8b(T)}\right)^{1/(\beta-1)}.$$

Then $E_x(T) < 0$ and x is intermediate.

Example 1. (Moore-Nehari) Consider

$$x'' + \frac{1}{4t^{(\beta+3)/2}}|x|^{\beta}\operatorname{sgn} x = 0 \qquad \beta > 1 \tag{7}$$

where $t \geq 1$. We have

$$F(t) = t^{(\beta+3)/2} b(t) = 1/4.$$

By Theorem 3 this equation has intermediate solutions such that

$$x(t) > 0, \quad x'(t) > 0 \quad t \geq 1. \tag{8}$$

One of them is

$$x(t) = \sqrt{t}.$$

Moreover, this equation has also oscillatory solutions, and subdominant solutions satisfying (8).

Example 2. Consider the equation

$$x'' + \frac{3}{16}\left(\frac{1}{t}\right)^{7/2} x^3(t) = 0 \quad (t \geq 1). \tag{9}$$

The function F is nonincreasing for $t \geq 1$.
By Theorem 3, equation (9) has intermediate solutions which are positive increasing on $[1, \infty)$.
One of them is

$$x(t) = t^{3/4},$$

and the energy function E_x is

$$E_x(t) = \left(\frac{9}{16} - \frac{3}{4} + \frac{3}{32}\right) t^{1/2} < 0$$

for any $t \geq 1$.

Existence of intermediate solutions of super-linear equation

$$\left(a(t)|x'|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \alpha < \beta \tag{E}$$

in case

$$\int_0^{\infty} a^{-1/\alpha}(s)ds < \infty, \quad \int_0^{\infty} b(s)ds = \infty.$$

$\hookrightarrow$ M. Marini, Z.D., *Positive decaying solutions for differential equations with phi-laplacian*, Bound. Value Probl. (2015).

Define

$$A(t) = \int_t^{\infty} a^{-1/\alpha}(s)ds.$$

Any eventually positive solution is decreasing:

$$\lim_{t\to\infty} x(t) = c_x, \quad 0 < c_x < \infty,$$

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} \frac{x(t)}{A(t)} = \infty,$$

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} \frac{x(t)}{A(t)} = c_x, \ 0 < c_x < \infty.$$

Intermediate solution=*slowly decaying solution*

Strongly decaying solution

Characteristic integrals:

$$Y = \int_1^\infty b(s) \left(\int_s^\infty a^{-1/\alpha}(r) dr \right)^\beta ds,$$
$$Z = \int_1^\infty \left(\frac{1}{a(s)} \int_1^s b(r) dr \right)^{1/\alpha} ds.$$

Equation has strongly decaying sols $\iff Y < \infty$.
Equation has sols tending to non-zero constant $\iff Z < \infty$.

Coexistence problem:

Theorem 4

Let $Y < \infty$ and $Z < \infty$. Then (E) *does not have intermediate solutions.*

Existence of intermediate solution:

Prototype in Case I:

$$x'' + b(t)|x|^{\beta}\,\mathrm{sgn}\,x = 0, \quad \beta > 1 \tag{1}$$

Prototype in Case II:

$$(t^{2\alpha}|x'|^{\alpha}\mathrm{sgn}\ x')' + b(t)|x|^{\beta}\mathrm{sgn}\ x = 0, \quad t \geq 1, \tag{E2}$$

where

$$b \text{ is differentiable on } [1,\infty),\ \ b(t) > 0$$

$$\int_1^{\infty} a^{-1/\alpha}(t)dt = \int_1^{\infty} t^{-2}(t)dt < \infty, \quad \int_1^{\infty} b(t)dt = \infty.$$

Example 3. Consider

$$\left(t^2x'\right)' + t^{4/3}x^3 = 0. \tag{10}$$

We have

$$A(t) = \int_t^\infty \frac{1}{s^2}\,ds = \frac{1}{t}, \quad Y < \infty, \quad Z = \infty.$$

Thus every nonoscillatory solution tends to zero.
Equation (10) has strongly decaying solutions, oscillatory solutions and solution

$$x(t) = \frac{\sqrt{2}}{3}t^{-2/3},$$

which satisfies

$$\lim_{t\to\infty}\frac{x(t)}{A(t)} = \lim_{t\to\infty} t^{1/3} = \infty,$$

i.e. x is slowly decaying solution.

... From continuous to discrete case:

$$\Delta^2 x_n + b_n |x_{n+1}|^\lambda \operatorname{sgn} x_{n+1} = 0, \quad \lambda \neq 1.$$

Physical applications:

↪ F. Weil (1980): Existence theorem for the difference equation $y_{n+1} - 2y_n + y_{n-1} = h^2 f(y_n)$

↪ R.B.Potts (1981): Exact solution of a difference approximation to Duffing's equation

J. W. Hooker, W.T. Patula (1983):

- Oscillation results are discrete analogues of the Atkinson and Belohorec oscillation criterion
- Differential equation has all oscillatory solutions bounded but difference equation has unbounded oscillatory solutions!
- Discrete analogue of Atkinson nonoscillation theorem turns out to be false! The corresponding difference equation has oscillatory solution!

How to explain discrepancies between continuous and discrete case?

Consider dynamical equation on a time scale T. To study the effect of the graininess to nonoscillation and asymptotic behavior.

Il piu nobile piacere ‘e la gioia del comprendere.

The noblest pleasure is the joy of understanding.

Leonardo da Vinci

Thank you for your attention!