Bolzano’s theorem for holomorphic mappings

Jean Mawhin

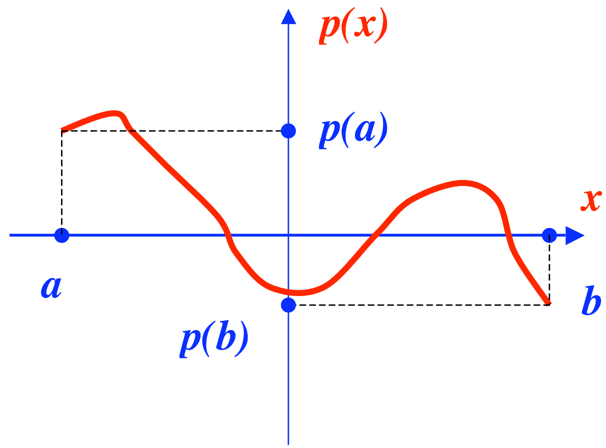
Université Catholique de Louvain

Cordially dedicated to Mauro Marini for his kindness and excellence

I. Some history

An old crossing principle

- $p(x) = \sum_{j=0}^{m} a_j x^j,\ m \geq 1,\ a_m \neq 0$
- **principle.** *If there exists two real numbers* $a < b$ *such that* $p(a)$ *and* $p(b)$ *have opposite signs,* p *vanishes between* a *et* b*.*



- **proof.** geometrical evidence

Real roots of numerical equations

- GIROLAMO CARDANO

1545 : *Ars Magnae,* Nuremberg, ch. XXX, *De regula Aurea*

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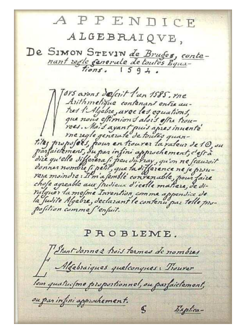
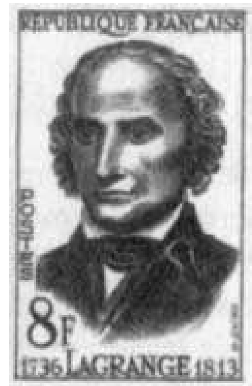
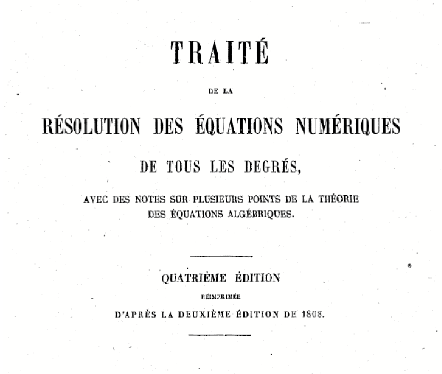


1594 : Appendice Algebraique, de Simon Stevin de Bruges, contenant regle generale de toutes Equations

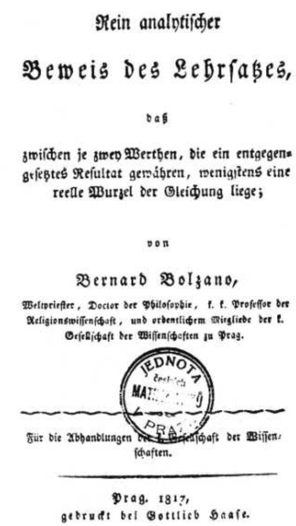
Joseph-Louis Lagrange




TRAITÉ
DE LA
RÉSOLUTION DES ÉQUATIONS NUMÉRIQUES
DE TOUS LES DEGRÉS,
AVEC DES NOTES SUR PLUSIEURS POINTS DE LA THÉORIE
DES ÉQUATIONS ALGÉBRIQUES.
QUATRIÈME ÉDITION
RÉIMPRIMÉE
D'APRÈS LA DEUXIÈME ÉDITION DE 1808.

- 1769 : *Sur la résolution des équations numériques,* Mémoires Acad. royale Sciences Belles-Lettres Berlin, 23
- 1795 : *Leçons élémentaires sur les mathématiques de l'École Normale,* J. École Polytechnique, Cahiers 7-8, 1812
- 1798 : *Traité de la résolution des équations numériques de tous les degrés,* Paris, (2^e éd. 1808) (2 1769 memoirs + 14 Notes)
- *several proofs (algebraic, geometrico-mechanical), all uncorrect*

Bernard Bolzano



Rein analytischer

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daß

zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von

Bernard Bolzano,

Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

Prag. 1817,
gedruckt bei Gottlieb Haase.

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  by BERNARD BOLZANO, *Secular priest, Doctor in Philosophy, Royal and Imperial Professor of Religious Science and Fellow of the Royal Society of Science in Prague*

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- **def.** *a function* $f(x)$ *varies according to the* **law of continuity** *for all the values of* x *located inside some bounds if,* x *being such an arbitrary value, the difference* $f(x+\omega)-f(x)$ *can be made smaller than any assigned quantity if one can always take* ω *as small as one wants*

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- **thm.** *If two positive functions of* x *,* $\psi(x)$ *and* $\varphi(x)$ *, vary according to the law of continuity for all the values of* x *located between* a *and* b *; if furthermore* $\psi(a)<\varphi(a)$ *and* $\psi(b)>\varphi(b)$ *; then there exists always some intermediate value of* x *between* a *and* b *for which* $\psi(x)=\varphi(x)$

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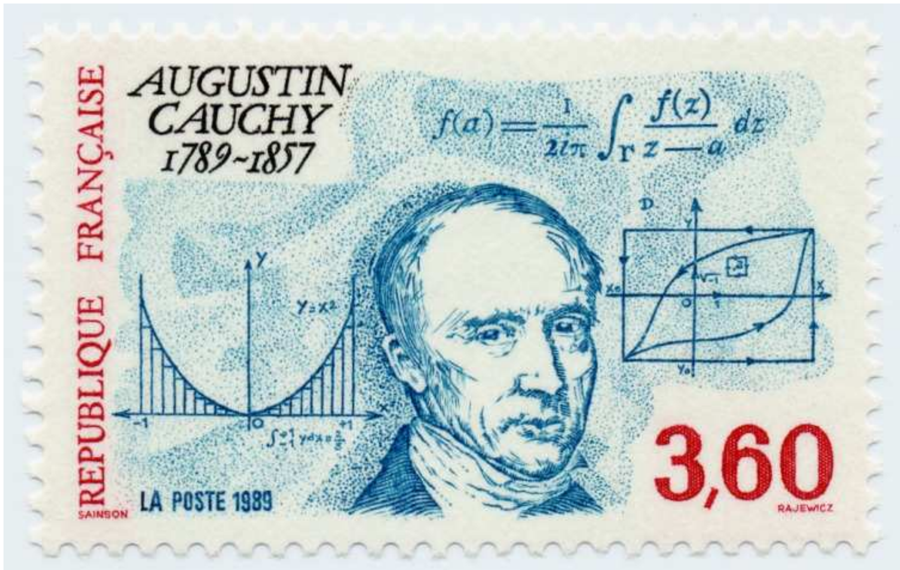
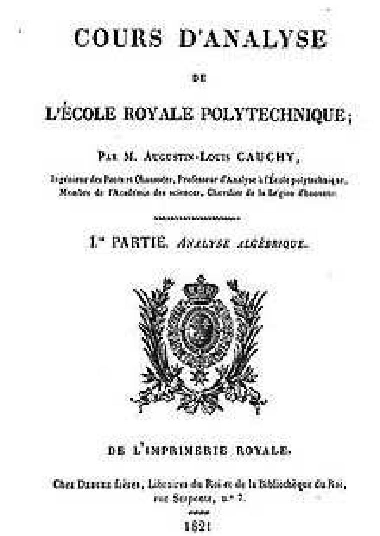
- **proof.** provides x as the supremum of the $y \in [a, b]$ such that que $\psi(t) < \varphi(t)$ on $[a, y]$
- only a rigorous theory of real numbers is missing

Augustin Cauchy





COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

I.re PARTIE. ANALYSE ALGÉBRIQUE.

DE L'IMPRIMERIE ROYALE.

1821

- 1821 *: Cours d'analyse de l'École royale polytechnique. Première partie. Analyse algébrique,* Paris, de Bure

  par M. AUGUSTIN CAUCHY, *Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur*

Intermediate value theorem

- **thm. (4 in ch. II)**. *If the function* $f(x)$ *is continuous between the limits* $x = a$ *and* $x = b$ *, and that one denotes by* c *any quantity between* $f(a)$ *and* $f(b)$ *, it will always be possible to satisfy equation* $f(x) = c$ *by one or several real values of* x *located between* a *and* b **(intermediate value property)**
- **proof.** geometrical (see analytical proof in Note III)

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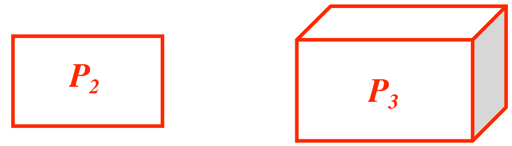
- **proof.** geometrical (see analytical proof in Note III)
- Note III. *On the numerical resolutions of equations.*
  thm. (1) *Let* $f(x)$ *be a real function of the variable* x *, which remains continuous with respect to this variable between the limits* $x = a, \quad x = b$ *. If the two quantities* $f(a)$ *,* $f(b)$ *have opposite sign, one will be able to satisfy equation* $f(x) = 0$ *by one or several real values of* x *located between* a *and* b
- **proof.** by bissection and nested intervals
- only missing is a theory of real numbers

Functions of several variables ?

- $P_n = [-a_1, a_1] \times \ldots \times [-a_n, a_n]$



- system of n equations in n unknowns

  $F_1(x_1, \ldots, x_n) = 0, \ \ldots \ , \ F_n(x_1, \ldots, x_n) = 0$

  with F_j continuous on P_n $(1 \leq j \leq n)$
- generalization of Bolzano's theorem ?

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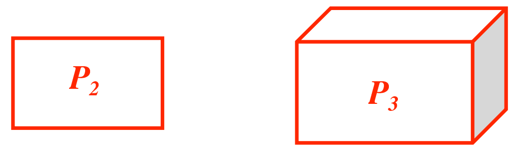

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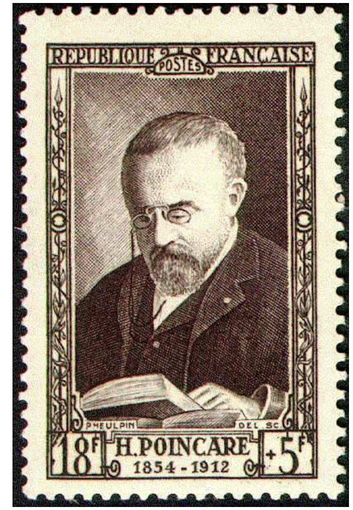
- generalization of Bolzano's theorem ?
- in modern terms : $F : P_n \to \mathbb{R}^n$ continuous
- replace sign condition on boundary points $-a_1, a_1$ of $P_1 = [-a_1, a_1]$ by sign conditions on the components F_j of F on **boundary faces** of P_n

  $P_n \cap \{x_j = -a_j\}, \;\; P_n \cap \{x_j = a_j\}$

Henri Poincaré



- 1883 : *Sur certaines solutions particulières du problème des trois corps,* Comptes Rendus Acad. Sciences Paris, *97, 251-252*
- 1884 *: Sur certaines solutions particulières du problème des trois corps,* Bulletin astronomique 1, 65-74

Statement and proof of Poincaré

- **thm.** *Let* $F_1, F_2, \ldots, F_n$ *be* n *continuous functions of* n *variables* $x_1, x_2, \ldots, x_n$ *;*
  the variable x_j *is assumed to stay between* $-a_j$ *and* a_j *.*
  Assume that, for $x_j = a_j$ *,* F_j *is constantly positive,*
  and for $x_j = -a_j$ *constantly negative;*
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- POINCARÉ's result almost immediately forgotten

A peculiar destiny for Poincaré’s thm

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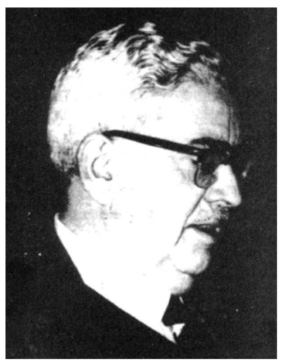
- 1974 : POINCARÉ's paper exhumed
  - statement renamed **Poincaré-Miranda** or **Bolzano-Poincaré thm**
  - direct proofs more sophisticated than for $n = 1$ (algebraic or differential topology, combinatorics, analysis)

French genious and Italian polemists

Henri Poincaré

Silvio Cinquini



Carlo Miranda

Giuseppe Scorza-Dragoni

Another n-dimensional generalization

- *Bolzano's assumptions on* $[-a, a] \;\Leftrightarrow\; xf(x) \geq 0$ *for* $|x| = a$
- $\mathbb{R}^n$: inner product $\langle\cdot|\cdot\rangle$, norm $\|\cdot\|$, $B_a = \{x \in \mathbb{R}^n : \|x\| \leq a\}$
- **thm.** $F : B_a \subset \mathbb{R}^n \to \mathbb{R}^n$ *continuous and such that* $\langle F(x)|x\rangle \geq 0$ *for* $\|x\| = a \;\Rightarrow\; F$ *has at least one zero in* B_a

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- 1911 : explicit statement in JACQUES HADAMARD, Appendix of vol. II of TANNERY's *Théorie des fonctions d'une variable*
  - called **Poincaré-Bohl's thm** by HADAMARD
  - **proof.** KRONECKER's index; application : BROUWER's FPT

JACQUES HADAMARD

A version for holomorphic functions

- 1982 : MAU-HSIANG SHIH, *Amer. Math. Monthly* 89
- $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ **holomorphic** on Ω : $f \in \mathcal{H}(\Omega, \mathbb{C})$

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- **thm.** $\Omega \supset 0$, *bounded domain,* $f \in C(\overline{\Omega}) \cap \mathcal{H}(\Omega, \mathbb{C})$,
  $\Re[\overline{z} f(z)] > 0$ *on* $\partial\Omega$ $\Rightarrow$ f *has a unique zero in* Ω
- **proof.** ROUCHÉ's thm applied to $f(z)$, $g(z) = \alpha z$,
  $\alpha = \inf_{z \in \partial\Omega} \Re[\overline{z} f(z)] / \sup_{z \in \partial\Omega} |z|^2$ on an approximation of Ω by squares

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- **rem.** $\Re[\overline{z} f(z)] = \Re z \cdot \Re f(z) + \Im z \cdot \Im f(z) = \langle (\Re z, \Im z) | (\Re f(z), \Im f(z)) \rangle$
  SHIH's condition = HADAMARD's condition on $\partial\Omega$

II. Bolzano's theorems for holomorphic functions : elementary proofs and uniqueness

Simple existence condition for a zero

- $\Omega \subset \mathbb{C}$ domain; $f \in \mathcal{H}(\Omega, \mathbb{C})$
- **piecewise C^k-cycle in** Ω : $\gamma \in C([a,b], \Omega)$ such that $\gamma(a) = \gamma(b)$ and there is a partition $a = a_0 < a_1 < a_2 < \ldots < a_q - 1 < a_q = b$ with $\gamma \in C^k([a_{j-1}, a_j], \Omega)\ (j = 1, \ldots, q)$
- $\int_\gamma f(z)\,dz := \int_a^b f[\gamma(t)]\gamma'(t)\,dt$
- **Cauchy's thm.** $\int_\gamma f(z)\,dz = 0$

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- **Cauchy's thm.** $\int_\gamma f(z)\,dz = 0$
- **thm (existence principle).** *Let* $g \in \mathcal{H}(\Omega, \mathbb{C})$, $\gamma : [a,b] \to \Omega$ *be a piecewise* C^2*-cycle reducible to a constant in* Ω*,* $\gamma([a,b]) = \partial\Delta$ *with* Δ *open, bounded,* $\overline{\Delta} \subset \Omega$*. If* $g(z) \neq 0$ *on* $\gamma([a,b])$ *and* $\int_\gamma \frac{dz}{g(z)} \neq 0$*, then* g *has at least one zero in* Δ

  proof. if not, $\int_\gamma \frac{dz}{g(z)} = 0$ by Cauchy's thm

Hadamard-Shih conditions on a circle

- **thm.** *if* $f \in \mathcal{H}(\Omega, \mathbb{C})$ *on some domain* $\Omega \supset B_a$ *and if* $\Re[\overline{z} f(z)] \geq 0 \, \forall \, z \in \partial B_a$, *then* f *has at least one zero in* $\overline{B}_a$

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- **proof.** sufficient to prove for $f_k(z) \equiv k^{-1} z + f(z) \; (k = 1, 2, \ldots)$
  - $\Re[\overline{z} f_k(z)] > 0 \; \forall \, z \in \partial B_a,$
  - $\gamma_a : [0, 2\pi] \to \Omega, \; t \mapsto a \exp(it)$
  - $\Im \left[\int_{\gamma_a} \frac{dz}{f_k(z)} \right] = \Im \left[\inf_{\gamma_a} \frac{z \overline{z}}{\overline{z} f_k(z)} \frac{dz}{z} \right] > 0$

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- **cor.** (**Brouwer FPT**) *if* $\Omega \subset B_a$, *any* $h \in \mathcal{H}(\Omega, \mathbb{C})$ *such that* $h(\partial B_a) \subset B_a$ *has at least one fixed point in* B_a

Poincaré-Miranda on a rectangle

- $P = [-a, a] \times [-b, b],\ P_{-a} = P \cap \{\Re z = -a\},\ P_a = P \cap \{\Re z = a\},\ P_{-b} = P \cap \{\Im z = -b\},\ P_b = P \cap \{\Im z = b\}$
- **thm.** *if* $f \in \mathcal{H}(\Omega, \mathbb{C})$ *on some domain* $\Omega \supset P,\ \Re f \leq 0$ *on* $P_{-a},\ \Re f \geq 0$ *on* $P_a, \Im f \leq 0$ *on* $P^{-b},\ \Im f \geq 0$ *on* P^b*, then* f *has at least one zero in* P

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- **proof.** sufficient to prove it for each $f_k(z) = k^{-1}z + f(z)$
  - inequalities on $\Re f_k,\ \Im f_k$ strict on ∂P
  - $\rho : [0, 4] \to \Omega$ with range ∂P
  - $\Im \left\{ \int_\rho \frac{dz}{f_k(z)} \right] = \Im \left[\int_\rho |f_k(z)|^{-2} [\Re f_k(z) - i \Im f_k(z)] \right\} > 0$

Poincaré-Miranda on a rectangle

- $P = [-a,a] \times [-b,b],\ P_{-a} = P \cap \{\Re z = -a\},\ P_a = P \cap \{\Re z = a\},\ P_{-b} = P \cap \{\Im z = -b\},\ P_b = P \cap \{\Im z = b\}$
- **thm.** *if* $f \in \mathcal{H}(\Omega, \mathbb{C})$ *on some domain* $\Omega \supset P,\ \Re f \le 0$ *on* $P_{-a},\ \Re f \ge 0$ *on* $P_a, \Im f \le 0$ *on* P^{-b}*,* $\Im f \ge 0$ *on* P^b*, then* f *has at least one zero in* P
- **proof.** sufficient to prove it for each $f_k(z) = k^{-1}z + f(z)$
  - inequalities on $\Re f_k, \Im f_k$ strict on ∂P
  - $\rho : [0,4] \to \Omega$ with range ∂P
  - $\Im \left\{ \int_\rho \frac{dz}{f_k(z)} \right] = \Im \left[\int_\rho |f_k(z)|^{-2} [\Re f_k(z) - i \Im f_k(z)] \right\} > 0$
- **cor.** *if* $\Re z \cdot \Re f(z) \ge 0$ *on* $P_{-a} \cup P_a$*,* $\Im z \cdot \Im f(z) \ge 0$ *on* $P^{-b} \cup P^b$*,* f *has at least one zero in* P
- **cor. Brouwer FPT** for a holomorphic function on a rectangle

Uniqueness

- $\Omega \subset \mathbb{C}$ a domain, $f \in \mathcal{H}(\Omega, \mathbb{C})$, Δ bounded domain with $\overline{\Delta} \subset \Omega$, γ a piecewise C^1-cycle which bounds Δ
- **thm. (argument principle).** *If* $f(z) \neq 0$ *on* $\gamma([a,b])$, f *has at most a finite number of zeros* $a_1, \ldots, a_p$ *in* Δ *and* $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^p m_j$, m_j *multiplicity of* a_j

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- **lem (invariance).** $F \in C(\Omega \times [0,1], \mathbb{C})$:
  - $F(z, \cdot) \in C^1([0,1], \mathbb{C}) \ \forall\, z \in \Omega$
  - $F(\cdot, \lambda) \in \mathcal{H}(\Omega, \mathbb{C}) \ \forall\, \lambda \in [0,1]$
  - $F(z,\lambda) \neq 0 \ \forall\, (z,\lambda) \in \gamma([a,b]) \times [0,1]$

  $\Rightarrow \ \lambda \mapsto \int_\gamma \frac{\partial_z F(z,\lambda)}{F(z,\lambda)}\, dz$ *is constant on* $[0,1]$

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  $\Rightarrow \ \lambda \mapsto \int_\gamma \frac{\partial_z F(z,\lambda)}{F(z,\lambda)}\, dz$ *is constant on* $[0,1]$
- **cor. (uniqueness)** $f(z) \neq 0$ *on* $\gamma([a,b])$, $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = 1$ $\Rightarrow$ f *has a unique zero in* Δ

Strict conditions on the boundary

- $\Omega \subset \mathbb{C}$ a domain, $f \in \mathcal{H}(\Omega, \mathbb{C})$, Δ bounded domain with $\overline{\Delta} \subset \Omega$, γ a piecewise C^1-cycle which bounds Δ
- **thm (Shih-Hadamard strict conditions).** *If* $\Re[\overline{z} f(z)] > 0$ $\forall\, z \in \partial\Delta$, f *has a unique zero in* Δ *and the zero is simple*
- **cor. (Brouwer FPT)** *If* $h(\partial\Delta) \subset \Delta$ *, unique fixed point in* Δ

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- **cor. (Brouwer FPT)** *If* $h(\partial\Delta) \subset \Delta$ *, unique fixed point in* Δ
- **thm (Poincaré-Miranda strict conditions).** *If* $P = [-a, a] \times [-b, b]$, $\Omega \supset P$, $f \in \mathcal{H}(\Omega, \mathbb{C})$ *is such that* $\Re f(z) < 0 \,\forall\, z \in P_{-a}$, $\Re f(z) > 0 \,\forall\, z \in P_a$, $\Im f(z) < 0 \,\forall\, z \in P^{-b}$, $\Im f(z) > 0 \,\forall\, z \in P^b$, *then* f *has a unique zero in int* P *, and this zero is simple*
- **proof.** $\frac{1}{2\pi i} \int_\rho \frac{f'(z)}{f(z)}\, dz = 1$ by invariance + uniqueness
- **cor. (Brouwer FPT)** *If* $h \in \mathcal{H}(\Omega, \mathbb{C})$ *on* $\Omega \supset P$ *and* $h(\partial P) \subset$ *int* P*, then* h *has a unique fixed point in int P*

III. Holomorphic maps in $\mathbb{C}^n$

Brouwer degree of holomorphic maps

- **def.** $f : \Omega \to \mathbb{C}^n$ is **holomorphic** in $\Omega \subset \mathbb{C}^n$ open, if $\forall\, a \in \Omega\,, \exists\, L_a : \mathbb{C}^n \to \mathbb{C}^n$ $\mathbb{C}$-linear : $\lim_{z\to a} \frac{\|f(z)-f(a)-L_a(z-a)\|}{\|z-a\|} = 0$

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- If $D \subset \Omega$ is open bounded and $0 \notin f(\partial D)$, the **Brouwer degree** $d_B[f, D, 0]$ generalizes $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ for $n \geq 1$
  - $d_B[f, D, 0]$ *is a nonnegative integer such that* $d_B[f, D, 0] > 0 \;\Leftrightarrow\; 0 \in f(D)$
  - $d_B[f, D, 0]$ *is greater or equal to the number of isolated zeros of* f *in* D
  - $d_B[f, D, 0] = 1 \;\Leftrightarrow\; f$ *has a unique zero* ζ *in* D *and* $J_f(\zeta) \neq 0$

Hadamard-Shih conditions for maps

- 1980 : MAU-HSIANG SHIH, *Proc. Amer. Math. Soc.* 79
- **thm.** *Let* $\Omega \subset \mathbb{C}^n$ *be open,* $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *,* D *be an open bounded neighborhood of* 0 *with* $\overline{D} \subset \Omega$ *, such that* $\sum_{j=1}^n \Re[\overline{z_j} f_j(z)] > 0, \;\; \forall\, z \in \partial D$ *. Then* f *has a unique zero in* D*, and this zero is non-degenerate*
- **proof.** simpler than SHIH's one

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- **proof.** simpler than SHIH's one
- existence survives under nonstrict inequalities in Hadamard-Shih conditions (approximation by strict maps)
- **cor. (Brouwer FPT)** *Let* $\Omega \subset \mathbb{C}^n$ *be open,* $h \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *, and* D *be open, bounded, convex such that* $\overline{D} \subset \Omega$.
  If $h(\partial D) \subset D$*,* h *has a unique fixed point in* D.
  If $h(\partial D) \subset \overline{D}$*,* h *has at least one fixed point in* $\overline{D}$

Poincaré-Miranda conditions for maps

- $P = \{z \in \mathbb{C}^n : \Re z_j \in [-a_j, a_j], \Im z_j \in [-b_j, b_j]$
  $(j = 1, \ldots, n)\}$
- **thm.** *Let* $\mathbb{C}^n \supset \Omega \supset P$ *be open and* $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *be such that*
  $\Re f_j(z) < 0, \ \forall\, z \in P \cap \{\Re z_j = -a_j\}$,
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  $\Im f_j(z) < 0, \ \forall\, z \in P \cap \{\Im z_j = -b_j\}$,
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  $\Im f_j(z) > 0, \ \forall\, z \in P \cap \{\Im z_j = b_j\} \ (j = 1, \ldots, n)$ *.*
  Then f *has a unique zero in int P , and this zero is not degenerate*
- existence of a zero in P survives if the Poincaré-Miranda inequalities are not strict
- **cor.** *Let* P *and* $\Omega \supset \overline{P}$ *be like above, and* $h \in \mathcal{H}(\Omega, \mathbb{C}^n)$ *.*
  If $h(\partial P) \subset$ *int* P*,* h *has a unique fixed point in int* P
  If $h(\partial P) \subset P$*,* h *has at least one fixed point in* P
- **references** : MAWHIN, *Chinese Annals of Mathematics*, to appear

Thank you for your kind attention and many happy years of retirement for Mauro !