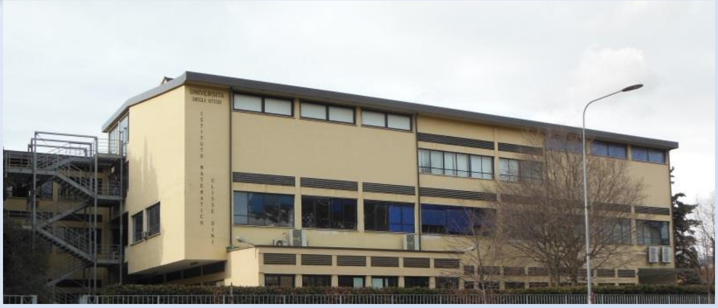
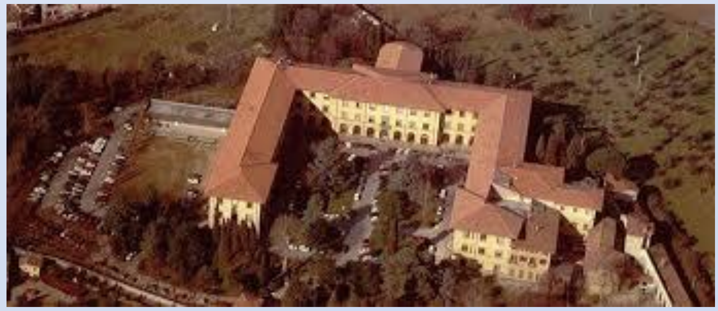



Workshop on

Old and New Trends in ODEs

Dedicated to the retirement of Prof. Mauro Marini

Florence,
2-3 December 2016

Main Speakers:

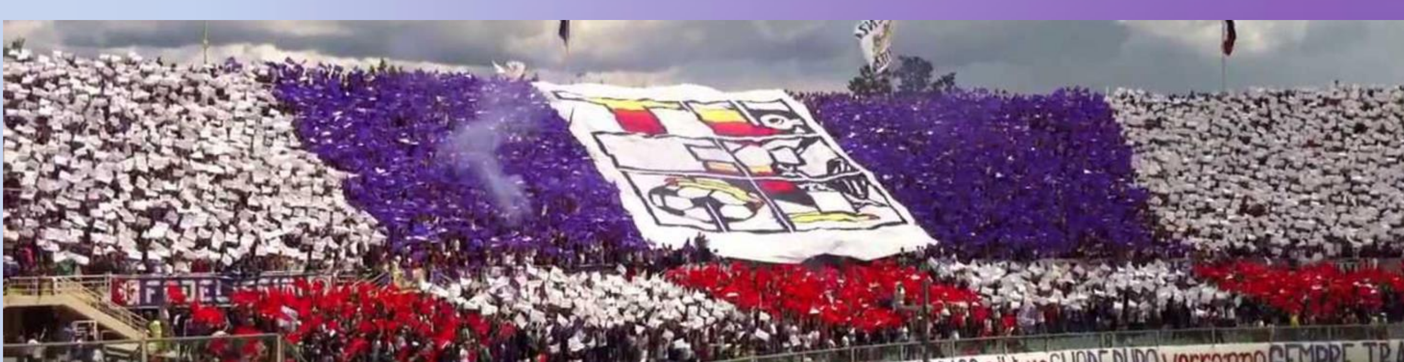
Gabriele Bonanno
Zuzana Došlá
Jean Mawhin
Józef Myjak
Fabio Zanolin

Scientific Committee:

Serena Matucci
Gabriele Villari

Organizing Committee:

All Mauro's friends

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Existence and global bifurcation of periodic solutions to functional differential equations with infinite delay

Maria Patrizia Pera

Firenze, December 2–3, 2016

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  one has

  $$f\colon \mathbb{R} \times BU((-\infty,0],M) \to \mathbb{R}^k$$

  such that

  $$f(t,\varphi) \in T_{\varphi(0)}M, \quad \forall (t,\varphi) \in \mathbb{R} \times BU((-\infty,0],M)$$

The equation $x'(t) = \lambda f(t, x_t)$

$BU((-\infty, 0], M)$ is a metric subspace of $BU((-\infty, 0], \mathbb{R}^k)$, which is Banach with the sup norm. The topology in $BU((-\infty, 0], M)$ is stronger than the compact open topology of $C((-\infty, 0], M)$

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- *the ODE's are obtained by setting*

$$f(t, \varphi) := \tilde{f}(t, \varphi(0))$$

- *the case of a finite delay $\tau > 0$ is obtained by setting*

$$f(t, \varphi) := \tilde{f}(t, \varphi(0), \varphi(-\tau))$$

Solutions to $x'(t) = \lambda f(t, x_t)$

Definition

Given $\lambda \geq 0$, a solution of (1) is a function $x\colon J \to M$ defined on an unbounded below interval J, such that $x_t \in BU((-\infty, 0], M)$ and there exists $\tau < \sup J$ for which

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Some references on RFDEs

- **infinite** delay in Euclidean spaces: Hale-Kato (1978), Hino-Murakami-Naito (1991), Oliva-Rocha (2010), Novo-Obaya-Sanchez (2007)

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- finite delay on **differentiable manifolds**: Oliva (1976)
- **infinite delay** on **differentiable manifolds** (existence, uniqueness, continuous dependence): Benevieri-Calamai-Furi-P. (2013), Discr. Contin. Dynam. Syst.

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For $\lambda = 0$, the T-periodic pair $(0, x)$ is such that x is constant, say $x(t) = p$, $\forall t$, and is said to be a trivial pair (denoted by $(0, p^-)$)

Definition

$p \in M$ is said to be a bifurcation point for (1) if any neighborhood of $(0, p^-)$ in $[0, +\infty) \times C_T(M)$ contains nontrivial T-periodic pairs (λ, x), i.e., with $\lambda > 0$.

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Let $w \colon M \to \mathbb{R}^k$ be the tangent vector field

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Theorem (necessary condition)

$p \in M$ *a bifurcation point* $\implies$ $w(p) = 0$

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Assume f locally Lipschitz in the second variable and sending bounded sets into bounded sets, M closed in $\mathbb{R}^k$ and U open in M. If $\deg(w, U) \neq 0 \implies$ there exists a global bifurcation branch, i.e. a connected set of nontrivial T-periodic pairs (λ, x), whose closure contains $(0, p^-)$, $p \in U$ and $w(p) = 0$, and

- i) *either is unbounded*
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Tools

- *fixed point index* for locally compact maps in metric ANRs (Granas, Nussbaum, Brown)

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- *fixed point index* for locally compact maps in metric ANRs (Granas, Nussbaum, Brown)
- *degree (also called Euler characteristic) of a tangent vector field* (Milnor, Hirsch, Guillemin-Pollack, Furi-P.)

Corollary

M compact with Euler-Poincaré characteristic $\chi(M) \neq 0 \implies$ there exists an unbounded (in λ) global bifurcation branch

Sketch of the proof

Set $U = M$. By the Poincaré-Hopf theorem

$$\deg(w, M) = \chi(M) \neq 0$$

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Theorem (Mawhin type continuation principle)

Let f be as above and U a relatively compact open subset of M. Assume that

i) *$w(p) \neq 0$ along the boundary ∂U of U and $\deg(w, U) \neq 0$;*

ii) *for any $\lambda \in (0, 1]$, the T-periodic orbits of $x'(t) = \lambda f(t, x_t)$ lying in $\overline{U}$ do not meet ∂U.*

Then, the equation $x'(t) = f(t, x_t)$ has a T-periodic orbit in U.

Second order RFDEs

Let us now give some ideas of second order RFDEs.
Consider the retarded motion equation on an m-dimensional smooth manifold $M \subseteq \mathbb{R}^k$

$$x''_\pi(t) = \lambda f(t, x_t), \quad \lambda \geq 0\,, \tag{2}$$

where

- $x''_\pi(t)$ is the tangential (or parallel) component of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t) \in M$.

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Aim: obtain global bifurcation results

Bifurcation branches to $x_\pi''(t) = \lambda f(t, x_t)$

Here branches are in

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Among the T-forced pairs we shall consider as *trivial* those of the type $(0, x)$ with x constant. (not all pairs $(0, x)$ are trivial!) Thus, if x is a T-periodic nonconstant geodesic, then the pair $(0, x)$ is **nontrivial**

- a nontrivial pair (λ, x) “close” to $(0, p^-)$ $\implies$ $\lambda > 0$

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Theorem (sufficient condition)

U open in M and $\deg(\bar{f}, U) \neq 0$ $\implies$ *there exists a global bifurcation branch*

(Benevieri-Calamai-Furi-P. (2014), J.Fixed Point Th. Appl.)

Bifurcation branches to $x''_\pi(t) = \lambda f(t, x_t)$

Corollary (Benevieri-Calamai-Furi-P., 2013, Adv.Nonlinear Stud.)

M compact with Euler-Poincaré characteristic $\chi(M) \neq 0 \implies$ there exists an unbounded global bifurcation branch

The branch might be bounded with respect to λ.
In particular,

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- The above equation is solvable in the case $M = S^2$ (spherical pendulum) and $M = S^{2n}$ (Benevieri-Calamai-Furi-P., 2011, J. Dyn. Diff. Eq.)
- Open problem even for ODEs

Comments (on the method used for second order equations)

A second order equation on a manifold M can be transformed into a first order system on the tangent bundle TM, but **not** of the type considered above, i.e. of the type

$$z'(t) = \lambda h(t, z_t).$$

In fact, the inertial equation

$$x''_\pi(t) = 0$$

does not correspond (in the phase space) to the equation

$$z'(t) = 0$$

For this reason second order equations cannot be handled with the techniques previously developed for first order ones